TOPOLOGY PRELIM EXAM

AUGUST 2017

1. Let $X = \mathbb{N} = \{1, 2, 3, ...\}$, and equip X with the topology

$$\tau = \{U \subset X \mid (2n-1) \in U \Rightarrow 2n \in U\}.$$

That is, $U \in \tau$ if and only if every odd number that is contained in U has its successor also contained in U.

(a) Prove that (X, τ) is not compact but it is locally compact (i.e. any point has a compact neighborhood).

(b) Determine (with proof) the connected components of (X, τ).

2. Let X, Y be topological spaces, D a dense subset of X and $f, g : X \to Y$ continuous maps such that $f(x) = g(x)$, for all $x \in D$. Show that if Y is Hausdorff, then $f = g$ on X.

3. Prove that there exists no one-to-one continuous map from $\mathbb{R}^n$ to $\mathbb{R}$ for $n > 1$.

Hint: How would such a map act on the unit sphere?

4. Let $\mathbb{D}$ be the closed unit disk, and $\mathbb{S}^1$ the unit circle.

(a) Prove that there is no retraction $r : \mathbb{D} \to \mathbb{S}^1$.

(b) Prove that every continuous map $f : \mathbb{D} \to \mathbb{D}$ has a fixed point.

5. Compute the fundamental group of the space obtained from two tori $\mathbb{S}^1 \times \mathbb{S}^1$ by identifying a circle $\mathbb{S}^1 \times \{x_0\}$ in one torus with the corresponding circle $\mathbb{S}^1 \times \{x_0\}$ in the other.

6. Let K be the Klein bottle and T the two-dimensional torus. Prove or disprove:

(a) There is a covering map from K to T.

(b) There is a covering map from T to K.