Real Analysis Preliminary Exam, August 2017

Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on Euclidean space is denoted by dx. By "measurable functions" we mean Lebesgue measurable functions.

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1. (15 points) State and prove the Monotone Convergence Theorem.

2. (15 points)

   (a) Write down the definition of a Stieltjes measure on the real line $\mathbb{R}$.

   (b) Find all Stieltjes measures $\nu \neq 0$ on $\mathbb{R}$ with

   $$\int_{\mathbb{R}} fg\, d\nu = \left(\int_{\mathbb{R}} f\, d\nu\right)\left(\int_{\mathbb{R}} g\, d\nu\right)$$

   for all non-negative continuous functions f and g.

3. (15 points) Prove or disprove three of the following statements.

   (a) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions which converges in $L^1(\mathbb{R}, dx)$ then it converges in measure.

   (b) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of integrable functions that converges almost everywhere on $[0,1]$, then it converges in $L^1([0,1], dx)$.

   (c) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions that converges almost everywhere on $[0,1]$, then it converges in $L^\infty([0,1], dx)$.

   (d) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions which converges in $L^1(\mathbb{R}, dx)$ then it converges almost everywhere.

4. (10 points) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let E be a measurable subset of $X \times Y$. Recall that the x-section of E is the set

   $$\{y \in Y \,|\, (x,y) \in E\}$$

   and the y-section of E is the set

   $$\{x \in X \,|\, (x,y) \in E\}.$$

   Use Fubini's theorem to prove that if the x-section of E has ν-measure 0 for μ-almost every $x \in X$, then the y-section of E has μ-measure 0 for ν-almost every $y \in Y$.

5. (10 points) Compute

   $$\lim_{n\to\infty} \int_{1/n}^{\infty} \frac{n^{3/2}y^{1/2} + y^{1/4}n^{1/4}}{n^2y^2 + n^{-1}}\, dy$$

   and justify all steps of your reasoning.

6. (10 points) Prove one of the following statements.

   (a) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are measurable then fg is also measurable.

   (b) Let μ be a signed measure. A set A is a null set with respect to μ if and only if $|\mu|(A) = 0$, where $|\mu| = \mu^+ + \mu^-$ is the total variation of μ.