Abstract Algebra Prelim Aug. 2018

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**)
   (a) (**5 pts**) In a finite abelian group, prove the order of each element divides the maximal order of all elements. (You may use the classification of finite abelian groups.)
   (b) (**5 pts**) In a field F, use part a to prove every finite subgroup of $F^\times = F - \{0\}$ is cyclic.

2. (**10 pts**) Let R be a commutative ring with identity and $R[x]$ be the polynomial ring over R.
   (a) (**3 pts**) Prove the ideal (x) in $R[x]$ is a prime ideal if and only if R is an integral domain.
   (b) (**4 pts**) Let I be an ideal of R. Prove that the following set is an ideal in $R[x]$:
   $$I[x] := \{f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x] \ : \ a_0, a_1, \ldots, a_n \in I\}.$$
   (c) (**3 pts**) Prove that an ideal I of R is a prime ideal if and only if the ideal $I[x]$ of $R[x]$ from part b is a prime ideal.

3. (**10 pts**) Let G be a group and H be a subgroup.
   (a) (**2 pts**) Define the normalizer of H in G.
   (b) (**4 pts**) Prove conjugate subgroups have conjugate normalizers: if N is the normalizer of H in G, then for each $g \in G$, gNg^{-1} is the normalizer of gHg^{-1} in G.
   (c) (**4 pts**) Let $G = \mathrm{GL}_2(\mathbf{R})$ and $H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbf{R}^\times \right\}$. Prove the normalizer of H in G is $\left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : a, b, c, d \in \mathbf{R}^\times \right\}$.

4. (**10 pts**) The Fibonacci numbers $\{f_n\}$ are determined recursively for $n \geq 0$ by $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 0$.
   (a) (**3pts**) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. For all integers $n \geq 1$, show $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$.
   (b) (**4pts**) Show the group $\mathrm{GL}_2(\mathbf{Z}/p\mathbf{Z})$, for prime p, has order $(p^2 - 1)(p^2 - p)$.
   (c) (**3pts**) Use parts a and b to help you find, with proof, *some* integer $n \geq 1$ such that $f_n \equiv 0 \bmod 10$ while $f_{n+1} \equiv 1 \bmod 10$. (Hint: Use the prime factorization of 10.)

5. (**10 pts**) An abelian group A is called *divisible* if, for each $a \in A$ and $n \in \mathbf{Z}^+$, there is a $b \in A$ (maybe not unique) such that $nb = a$. (Formally it says we can "divide" a by n, but the choice may not be unique so do *not* write $b = \frac{1}{n}a$.) For example, $(\mathbf{R}, +)$ is divisible. Also $(\mathbf{C}^\times, \times)$ is divisible since, for all $n \in \mathbf{Z}^+$, a number in $\mathbf{C}^\times$ has an nth root in $\mathbf{C}^\times$ (not unique if $n > 1$).

   Prove a nonzero divisible group can't be finitely generated.

6. (**10 pts**) Give examples as requested, with justification.
   (a) (**2.5 pts**) Two nonconjugate elements of S_4 that have the same order.
   (b) (**2.5 pts**) Two commutative rings that *are not* isomorphic as rings but *are* isomorphic as additive groups.
   (c) (**2.5 pts**) A formula for a ring isomorphism $\mathbf{R}[x]/(x^2 - 1) \to \mathbf{R} \times \mathbf{R}$.
   (d) (**2.5 pts**) A cyclic $\mathbf{Z}[x]$-module (this means a $\mathbf{Z}[x]$-module having one generator) with a submodule that is not cyclic.