Applied Math Prelim, August 2018

(1) Let H be a Hilbert space.

(a) Let $\{w_n\}_{n=1}^{\infty}$ be an orthogonal set. Show that $\sum_{n=1}^{\infty} w_n$ converges if and only if $\sum_{n=1}^{\infty} \|w_n\|^2 < \infty$. Moreover $\| \sum_{n=1}^{\infty} w_n\|^2 = \sum_{n=1}^{\infty} \|w_n\|^2$.

(b) Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal basis in H. Suppose $\{v_n\}_{n=1}^{\infty}$ is an orthonormal set. If $\sum_i \|u_i - v_i\|^2 < 1$, show that $\{v_n\}_{n=1}^{\infty}$ is an orthonormal basis.

(hint: Recall that $\{v_n\}_{n=1}^{\infty} \subset H$ is an orthonormal basis if $w \in H$ and $w \perp v_i$ for all i imply $w = 0$.)

(2a) Find the Green's function $G(x, y)$ for the operator A where

$$Au \equiv u'' - u$$

with $u(0) = u(1) = 0$.

(2b) Find all the eigenvalues and eigenfunctions of the operator A.

(2c) Define $T : L^2(0,1) \to L^2(0,1)$ such that for any $f \in L^2(0, \pi)$,

$$(Tf)(x) = \int_0^1 G(x,y) f(y)\, dy \ .$$

Explain what spectral theorem is and why it is applicable.

(2d) Let $f \in C[0,1]$ and $u = Tf$. Show that $u(0) = u(1) = 0$ and evaluate $u'(0)$ in term of f.

(3a) Let X and Y be Hilbert spaces and $F : X \to Y$. Give the definition for F being Frechet differentiable at $u_0 \in X$.

(3b) Define $U \equiv \{w \in C^2[0,1] : w(0) = w(1) = 0\}$ and consider $\mathcal{I} : U \to \mathbb{R}$ such that for any $u \in W$

$$\mathcal{I}(u) \equiv \int_0^1 (\frac{1}{2} u'^2 + \frac{1}{2} u^2 - u)\, dx \ .$$

Show that $\mathcal{I}$ is Frechet differentiable at any $u \in U$ and find the derivative $\mathcal{I}'(u)$.

(3c) Find a critical point u_1 of $\mathcal{I}$, i.e. $\mathcal{I}'(u_1) w = 0$ for any $w \in U$.

(4a) Let T be a distribution. Give the definition for its distributional derivative ∂T. Show that ∂T is also a distribution.
(4b) Define $u : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = \begin{cases} \sinh x, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

Let $\tilde{u}$ be the distribution defined by $\tilde{u}(\phi) \equiv \int_{-\infty}^{\infty} u(x)\,\phi(x)\,dx$ for all $\phi \in \mathcal{D}$. Show that $\partial^2\tilde{u} - \tilde{u} = \delta$. Here δ is the delta distribution.
(4c) Suggest a different function w such that $\partial^2\tilde{w} - \tilde{w} = \delta$. Is there a function w that goes to 0 as $|x| \to \infty$?

(5a) Let X be a Hilbert space and $A_n : X \to X$ be a linear bounded compact operator for all $n = 1, 2, \ldots$. Suppose $A_n \to B$ in operator norm for some linear bounded operator $B : X \to X$. Show that B is a compact operator as well.

(5b) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in the Hilbert space X. Suppose $B : X \to X$ is a linear bounded operator with $\sum_{j=1}^{\infty} \|Be_j\|^2 < \infty$. (This is known as the Hilbert-Schmidt operator). For any $w \in X$, define $A_n : X \to X$ such that $A_n w = \sum_{j=1}^{n} \langle w, e_j \rangle Be_j$. Show that A_n is a linear bounded compact operator and $A_n \to B$ in operator norm.

(5c) Given $G : (0,1) \times (0,1) \to \mathbb{R}$ satisfying $\int_0^1 \int_0^1 |G(x,y)|^2\,dx\,dy < \infty$. For any $u \in L^2(0,1)$ define $B : L^2(0,1) \to L^2(0,1)$ such that $Bu\,(x) = \int_0^1 G(x,y)u(y)\,dy$. Suppose $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of L^2, use the Parseal's relation or otherwise to show $\int_0^1 \int_0^1 |G(x,y)|^2\,dx\,dy = \sum_{j=1}^{\infty} \|Be_j\|^2$. i.e. B is a Hilbert-Schmidt operator.