COMPLEX ANALYSIS PRELIM

AUGUST, 2018

Notation and conventions:

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.
- The terminology *analytic* function and *holomorphic* function may be used interchangeably.

Problem 1. How many roots (counted with multiplicity) does the function $f(z) = 5z^3 + e^z + 1$ have in the unit disk $\mathbb{D}$?

Problem 2. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2} \qquad \text{for all } z \text{ in } \mathbb{D}.$$

Problem 3. Let a be a real number. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function that satisfies

$$\int_0^{2\pi} |f(re^{it})| \, dt \le r^a$$

for all $r > 0$. Show that f is a polynomial.

Problem 4. Let $\Omega = \{z = re^{i\theta} \in \mathbb{C} : 0 < r < 2 \text{ and } 0 < \theta < 3\pi/2\}$. Explicitly describe a one-to-one holomorphic map from Ω onto the unit disk $\mathbb{D}$.

Problem 5. Let f be a holomorphic function on $\mathbb{D}$. Suppose $|f(z)| \le |f(z^2)|$ for all $z \in \mathbb{D}$. Show that f is a constant.

Problem 6.

(1) Let $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, oriented in the counter-clockwise direction. Evaluate

$$\int_\gamma \frac{z^2 + 1}{z(z^2 + 4z + 1)} \, dz.$$

(2) Evaluate

$$\int_0^{2\pi} \frac{\cos x}{2 + \cos x} \, dx.$$

(Hint: You can use the contour integral in part (1).)

Problem 7. Suppose that $f(z)$ is holomorphic on the punctured unit disk $\mathbb{D} \setminus \{0\}$ and that the real part of f is positive. Prove that f has a removable singularity at 0.