TOPOLOGY PRELIM, AUGUST 2018

Convention

- Let A and B be two sets. We denote $A \setminus B = \{x \in A; x \notin B\}$.
- Unless otherwise indicated, $\mathbb{R}^n$ is endowed with the standard topology.

1. Let Γ be a subset in a compact topological space such that every point of Γ is an isolated point of Γ. Is Γ necessarily a finite set? Prove your assertion.

2. Compute the fundamental group of the quotient space $(S^1 \times S^1)/(S^1 \times \{x\})$, where x is a point in S^1.

3. Let $\mathcal{Z}$ be the topology on $\mathbb{R}^2$ such that every nonempty open set of $\mathcal{Z}$ is of the form $\mathbb{R}^2 \setminus \{\text{at most finitely many points}\}$.
   (i) Is $(\mathbb{R}^2, \mathcal{Z})$ Hausdorff? Prove your assertion.
   (ii) Is $(\mathbb{R}^2, \mathcal{Z})$ first countable? Prove your assertion.

4. Show that every continuous map from $\mathbb{RP}^2$ to S^1 is homotopic to a constant map.

5. Let M be the quotient space of $\mathbb{R}^3 \setminus \{0\}$ obtained by identifying the points (x, y, z) with $(2^m x, 2^m y, 2^m z)$ for any integer m. Is M homeomorphic to $S^2 \times S^1$? Prove your assertion.

6. Let X be a topological space and $\pi : \mathbb{R}^2 \to X$ a covering map. Let $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and let K be a compact subset of X.
   (i) Suppose $\pi : \mathbb{R}^2 \setminus B \to X \setminus K$ is a homeomorphism. Show that X is homeomorphic to $\mathbb{R}^2$.
   (ii) Suppose $\mathbb{R}^2 \setminus B$ is homeomorphic to $X \setminus K$, but the homeomorphism may not be given by π. Is X necessarily homeomorphic to $\mathbb{R}^2$? Prove your assertion.