Real Analysis Prelim

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Instructions: Do as many of the following problems as you can. Four completely correct solutions will guarantee a PhD pass. A few completely correct solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill in the gap. You may use any standard theorem from the real analysis course, identifying it either by name or stating it in full.

Notation and Conventions:

- $(\mathbb{R}^n, \mathcal{L}, m)$ denotes the standard Lebesgue measure space on $\mathbb{R}^n$. As usual, integration with respect to Lebesgue measure may be denoted by $\int f(x)\,dm(x)$ or $\int f(x)\,dx$ or a similar variant.
- $B(x,r)$ denotes the open ball in $\mathbb{R}^n$ with center x and radius r.

Problems:

1. Let X be a nonempty set and let μ^* be an outer measure on X. Recall that an outer measure is a function $\mu^* : 2^X \to [0,\infty]$ satisfying $\mu^*(\emptyset) = 0$, $\mu^*(E) \le \mu^*(F)$ whenever $E \subset F$, and $\mu^*(\bigcup_{n=1}^\infty E_n) \le \sum_{n=1}^\infty \mu^*(E_n)$ for any sequence of sets $E_1, E_2, \ldots$. A set $A \subset X$ is called μ^*-*measurable* if
$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \quad \text{for all } E \subset X. \tag{P1}$$
Let $\mathcal{M}$ denote the family of μ^*-measurable sets. Prove that $\mathcal{M}$ is a σ-algebra on X and $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure on X.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) Prove that if $f_n, g_n, f, g \in L^1(X,\mu)$, $|f_n| \le g_n$ for all n, $f_n \to f$ μ-a.e., $g_n \to g$ μ-a.e., and $\displaystyle\int g_n\,d\mu \to \int g\,d\mu$, then $\displaystyle\int f_n\,d\mu \to \int f\,d\mu$.
   (b) Let $1 \le p < \infty$. Suppose $f_n, f \in L^p(X,\mu)$ and $f_n \to f$ μ-a.e. Prove that $\displaystyle\int |f_n - f|^p\,d\mu \to 0$ if and only if $\displaystyle\int |f_n|^p\,d\mu \to \int |f|^p\,d\mu$.

3. Let T be the triangle $\{(x,y) \in \mathbb{R}^2 : 0 \le |x| \le y \le 1\}$, and let μ be the restriction of Lebesgue measure to T. Prove that if $f \in L^2(T,\mu)$, then
   (a) $f \in L^1(T,\mu)$ and
   (b) $\liminf_{y\to 0+} \displaystyle\int_{-y}^{y} |f(x,y)|\,dx = 0$.

4. (a) Let $U \subset \mathbb{R}^n$ be an open set and let $N \subset \mathbb{R}^n$ be a Lebesgue null set. Prove that

$$\lim_{r \to 0+} \frac{m(B(x,r) \setminus (U \setminus N))}{r^n} = 0 \quad \text{for all } x \in U \setminus N. \tag{P4a}$$

(b) Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set. Prove that

$$\lim_{r \to 0+} \frac{m(B(x,r) \setminus A)}{r^n} = 0 \quad \text{for } m\text{-a.e. } x \in A. \tag{P4b}$$

5. Show that there exists a Borel measure ν on $\mathbb{R}$ that satisfies both

   i. ν and m are mutually singular; and,

   ii. $0 < \nu(B(x,r)) < \infty$ for all $x \in \mathbb{R}$ and $r > 0$.

6. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then

$$\int_{\mathbb{R}} f(x)^4 \, dx = 4 \int_0^\infty m(\{x \in \mathbb{R} : |f(x)| > t\}) \, t^3 \, dt. \tag{P6}$$

7. Let $\mathcal{F}$ denote the family of all continuous functions $u : \mathbb{R}^n \to \mathbb{R}$ that satisfy

$$u(x) = \frac{1}{m(B(x,1))} \int_{B(x,1)} u(y) \, dy \quad \text{for all } x \in \mathbb{R}^n. \tag{P7}$$

Show that if $\{u_k\}_{k=1}^\infty \subset \mathcal{F}$ is uniformly bounded on each compact subset of $\mathbb{R}^n$, then there is a subsequence of $\{u_k\}$ that converges uniformly on each compact subset of $\mathbb{R}^n$ to a function $u \in \mathcal{F}$.