

**Risk Theory Prelims for Actuarial Students**  
**Monday, 20 August 2018**  
**MONT 313, 9:00 am - 1:00 pm**

**Instructions:**

1. There are five (5) questions here and you are to answer all five. Each question is worth 20 points.
2. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
3. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

**Question No. 1:**

For any given insured from a portfolio of 100,000 policies, the claim frequency variable  $N$  follows a Poisson distribution with mean parameter  $\lambda$  which varies among the insured. Indeed, it has been found that  $\lambda$  is distributed as the sum of 5 identically and independently distributed exponential variables, each with mean 10.

Suppose the portfolio increases to 500,000 policies and in a manner such that all risk characteristics per insured stay the same. Let  $N^*$  be the claim frequency for a given insured from this new portfolio.

- (a) Give an expression for the moment generating function of  $N^*$ .
- (b) Calculate the mean of  $N^*$ .
- (c) Calculate the variance of  $N^*$ .

**Question No. 2:**

For a given random variable  $X$ , define the conditional tail expectation as

$$\text{CTE}_X(x_q) = E(X|X > x_q),$$

for  $0 < q < 1$ , where  $x_q$  is the  $q$ -th quantile of the distribution of  $X$  defined as  $x_q = \inf\{x|F_X(x) \geq q\}$ .  $F_x(\cdot)$  is the distribution function of  $X$ .

Show that for a normal random variable  $X$  with mean  $\mu$  and variance  $\sigma^2$ , the conditional tail expectation can be written as

$$\text{CTE}_X(x_q) = \mu + h\sigma^2,$$

where  $h$  can be expressed as

$$h = \frac{1}{\sigma} \frac{\phi(k)}{\Phi(k)},$$

for a constant  $k$  and where  $\phi(\cdot)$  and  $\Phi(\cdot)$  are the density and distribution functions of a standard normal variable, respectively. Give an expression for  $k$ .

**Question No. 3:**

In a collective risk model where the aggregate claim is defined by  $S = X_1 + X_2 + \cdots + X_N$ , you are given:

- (i) Claim frequency  $N$  has a Poisson distribution with mean 4.
  - (ii) Claim amount  $X$  has the distribution  $p(1) = 0.5$ ,  $p(2) = 0.3$ , and  $p(3) = 0.2$ .
- (a) Use the Panjer's recursion formula to show that

$$\Pr(S = n) = \frac{1}{n} \times [k_1 \Pr(S = n - 1) + k_2 \Pr(S = n - 2) + k_3 \Pr(S = n - 3)],$$

for constants  $k_1$ ,  $k_2$ , and  $k_3$ . Determine the values of these constants.

- (b) Calculate  $E[(S - 3)_+]$ .

**Question No. 4:**

Individual loss amount  $X$  follows a two-parameter Weibull distribution with mean 10 and variance 500. An insurance policy on  $X$  has a deductible amount of 2 and a policy limit of 100 per loss.

Assume loss amount increased due to inflation by 5% uniformly.

- (a) Show that  $\tau = 1/2$  and determine the value of the parameter  $\theta$ .
- (a) Calculate the expected value of claims per loss after the inflation. You may leave your answer in terms of the incomplete gamma function.
- (b) Calculate the variance of claims per loss after the inflation. You may leave your answer in terms of the incomplete gamma function.

**Question No. 5:**

Consider a claims random variable  $X$  such that given a risk class parameter  $\gamma$ , the random variable  $X|\gamma$  has an exponential distribution with mean parameter  $1/\gamma$ . The risk class parameter

$\gamma$  has a gamma( $\alpha, \lambda$ ) distribution with density  $f(\gamma) = \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma}$ .

- (a) Show that the unconditional distribution function of  $X$  is given by  $F(x) = 1 - (1 + x/\lambda)^{-\alpha}$ .
- (b) Suppose, conditional on the risk class  $\gamma$ , that  $X_1$  and  $X_2$  are independent and identically distributed. Assume that they both come from the same risk class  $\gamma$  that induces a dependency. Show that the joint distribution function of  $(X_1, X_2)$  can be expressed as

$$\begin{aligned} F(x_1, x_2) &= \Pr(X_1 \leq x_1, X_2 \leq x_2) \\ &= F(x_1) + F(x_2) - 1 + [(1 - F(x_1))^{-1/\alpha} + (1 - F(x_2))^{-1/\alpha} - 1]^{-\alpha}. \end{aligned}$$

— end of exam —

## APPENDIX

A random variable  $X$  is said to have a two-parameter Weibull distribution if its density has the form

$$f(x) = \frac{1}{x} \tau (x/\theta)^\tau e^{-(x/\theta)^\tau}, \quad \text{for } x > 0.$$

This distribution satisfies the following:

$$E[X^k] = \theta^k \Gamma(1 + k/\tau), \quad \text{for any } k > -\tau.$$

and

$$E[(X \wedge x)^k] = \theta^k \Gamma(1 + k/\tau) \Gamma[1 + k/\tau; (x/\theta)^\tau] + x^k e^{-(x/\theta)^\tau}, \quad \text{for any } k > -\tau.$$

A discrete random variable  $N$  is said to belong to the  $(a, b, 0)$  class of distributions if it satisfies the relation

$$\Pr(N = k) = p_k = \left(a + \frac{b}{k}\right) \cdot p_{k-1}, \quad \text{for } k = 1, 2, \dots,$$

for some constants  $a$  and  $b$ . The initial value  $p_0$  is determined so that  $\sum_{k=0}^{\infty} p_k = 1$ .