Preliminary Exam **Math 5310** Winter, 2009

Name:____________________________ Section:__________

Important !! Do (1) through (4) and **either** (5) or (6). Clearly indicate which of these you chose.

1. The dictionary order is a total order on R^2 ($(a,b) < (c,d)$ if either $a < c$ or $a = c$ and $b < d$). Let X be the set R^2 with the order topology relative to the dictionary order.

a) Prove or disprove: X is homeomorphic to the real line R.
b) Determine the components of X

2. Let R_l denote the real line with the lower limit topology (with a basis consisting of sets of the form $[a, b)$).

a) Prove or disprove: X is second countable.
b) Determine the components of $R_l \times R_l$.

3. Let $p : X \longrightarrow Y$ be a quotient map. Prove or disprove:

a) If X is Hausdorff, then Y is Hausdorff.
b) If Y is Hausdorff, then X is Hausdorff.

4. Let $A \subset X$, where X is a topological space. Denote by $Int(A)$ and $Bd(A)$ the interior and boundary of A, respectively.

a) Prove or disprove: If A is connected, then $Int(A)$ is connected.
b) Prove or disprove: If both $Int(A)$ and $Bd(A)$ are connected, then A is connected.

5. Let X be a topological space.

i) X is **countably compact** if every countable open cover has a finite subcover,
ii) X has the Bolzano-Weierstrass property if every infinite subset of X has a cluster point.
iii) X has the **Cantor intersection property** if every telescoping sequence of non-empty closed subsets, $F_1 \supset F_2 \supset \cdots \supset F_i \supset \cdots$, has a non-empty intersection, $\cap F_i \neq \emptyset : i = 1, 2, \cdots$.

Show that if X is a T_1 space, (i), (ii) and (iii) are equivalent.

6. A Hausdorff topological space is called "paracompact" if every open cover of X has a locally finite refinement.

Remark : Given a cover $\{U_\alpha,\, \alpha \in A\}$ of X, a cover $\{V_\beta,\, \beta \in B\}$ of X is called a locally finite refinement of $\{U_\alpha,\, \alpha \in A\}$ if (i) any element of $\{V_\beta,\, \beta \in B\}$ is contained in some element of $\{U_\alpha,\, \alpha \in A\}$ and (ii)

for any point $x \in X$, there exists a neighborhood U of x such that U meets at most a finite number of elements in $\{V_\beta,\, \beta \in B\}$.

Prove that every paracompact space is normal.