MATH 5510(313)   January 2009   PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer three out of five questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Let $x_0, x_1, x_2, \ldots, x_n$ (such that $x_k \neq x_m$ when $k \neq m$) be given. Let

$$L_k(x) = \begin{cases} \frac{(x-x_1)\cdot\ldots\cdot(x-x_n)}{(x_0-x_1)\cdot\ldots\cdot(x_0-x_n)} & k = 0 \\ \frac{(x-x_0)\cdot\ldots(x-x_{k-1})(x-x_{k+1})\cdot\ldots\cdot(x-x_n)}{(x_k-x_0)\cdot\ldots(x_k-x_{k-1})(x_k-x_{k+1})\cdot\ldots\cdot(x_k-x_n)} & 0 < k < n \\ \frac{(x-x_0)\cdot\ldots\cdot(x-x_{n-1})}{(x_n-x_0)\cdot\ldots\cdot(x_n-x_{n-1})} & k = n \end{cases}$$

Prove that for $k = 0, 1, \ldots, n$ we have

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\ x_0^3 & x_1^3 & x_2^3 & \cdots & x_n^3 \\ \vdots & \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & x_2^n & \cdots & x_n^n \end{bmatrix}}_{\text{Vandermonde matrix}} \begin{bmatrix} L_0(x) \\ L_1(x) \\ L_2(x) \\ L_3(x) \\ \cdots \\ L_n(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \cdots \\ x^n \end{bmatrix}$$

(b) Use the condition

$$x_i \neq x_j \quad for \quad i \neq j,$$

to prove that the above Vandermonde matrix is nonsingular.

(c) Let $P_{i_0 i_1 \ldots i_k}(x)$ be the (unique) polynomial that interpolates at points

$$(x_{i_m}, f_{i_m}) \qquad m = 0, \ldots k.$$

Prove that these polynomials are linked by the following recursion:

$$P_{i_0 \ldots i_k}(x) = \frac{(x - x_{i_0}) P_{i_1 i_2 \ldots i_k} - (x - x_{i_k}) P_{i_0 i_1 \ldots i_{k-1}}}{x_{i_k} - x_{i_0}}.$$

(d) Use the result of (c) to formulate and to derive the Neville algorithm for evaluating the interpolation polynomial $P_{0,1,\ldots,n}(x)$ at a point x, given the interpolation data $\{x_i, f_i\}_{i=0}^n$.

Q2) (a) Let

$$y = (c - \sum_{i=1}^{k-1} a_i b_i)/b_k$$

is evaluated in the standard model of floating point arithmetic according to

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s = c
for     i = 1 : k - 1
        s = s - a_i b_i
end
y = s/b_k
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Prove that computed $\widehat{y}$ satisfies

$$b_k\widehat{y}(1+\theta_k) = c - \sum_{i=1}^{k-1} a_i b_i (1+\theta_i)$$

with $|\theta_i| \le \gamma_i := \frac{iu}{1-iu}$ where u is the machine precision.

(b) Use the above result to show that if the Gaussian elimination algorithm applied to an $n \times n$ matrix A runs to completion, the computed factors $\widehat{L}$ and $\widehat{U}$ satisfy

$$\widehat{L}\widehat{U} = A + \Delta A$$

with

$$|\Delta A| \le \gamma_n |\widehat{L}| \cdot |\widehat{U}|.$$

Q3) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

(a) Define the DFT matrix and derive the formula for its inverse.

(b) Describe the FFT algorithm for arbitrary $N = 2^k$. Specifically, describe the divide-and-conquer strategy and provide the formula reducing F_N to two $F_{N/2}$'s.

(c) Let $C(N)$ denote the cost of the FFT of the order N. Prove the formula

$$C(N) = \begin{cases} b & N = 1 \\ 2C(\frac{N}{2}) + bN & N > 1 \end{cases}$$

(d) Use the result of (c) to derive the assymptotic formula (i.e., up to a multiplicative constant) for the number of arithmetic used by the FFT algorithm.

(e) Define a circulant matrix and the DFT matrix. Prove that any circulant is diagonalized by the DFT matrix.

Q4) Let $w(x)$ be a positive continuous function on $[a, b]$. For $j = 1, 2, ...,$ let $p_j(x)$ be the corresponding monic orthogonal polynomial of degree j, i.e.,

$$p_j(x) = x^j + a_1 x^{j-1} + \cdots + a_j,$$

such that $(p_j, p_k) = \int_a^b w(x)p_j(x)p_k(x)dx = 0$ if $j \neq k$. In particular $p_0(x) = 1$.

(a) Prove that the roots $x_1, .., x_n$ of $p_n(x)$ are real, simple and lie in (a, b).

(b) Prove that $p_n(x)$ satisfy a three term recurrence relation, i.e.,

$$p_{i+1}(x) = (x - \delta_{i+1})p_i(x) - \gamma_{i+1}^2 p_{i-1}(x), \quad \imath \ge 0,$$

where $p_{i-1} = 0, \quad \gamma_1 = 0$, and

$$\delta_{i+1} = \frac{(xp_i, p_i)}{(p_i, p_i)}, \quad i \ge 0, \quad \gamma_{i+1}^2 = \frac{(p_i, p_i)}{(p_{i-1}, p_{i-1})}, \quad i \ge 1.$$

(c) For $a = -1; b = 1; w(x) = 1$; find $p_1(x)$ and $p_2(x)$.

Q5) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Define a Hankel matrix. Let H be an $n \times n$ positive definite Hankel matrix. Relate the factorization

$$H\widetilde{U} = \widetilde{L} \tag{1}$$

to the standard LDL^* factorization of H to prove that (1) always exists and it is unique. Here $\widetilde{U}$ is a unit (i.e., with 1's on the main diagonal) upper triangular matrix, and $\widetilde{L}$ is a lower triangular matrix.

(b) Let $\langle \cdot, \cdot \rangle$ be an inner product in the vector space Π_n (of all polynomials whose degree does not exceed n). Let the above Hankel matrix H be a moment matrix, i.e., $H = [\langle x^i, x^j \rangle]_{i,j=0}^n$. Let

$$u_k(x) = u_{0,k} + u_{1,k}x + u_{2,k}x^2 + \ldots + u_{k-1,k}x^{k-1} + x^k. \tag{2}$$

be the k-th orthogonal polynomial with respect to $\langle \cdot, \cdot \rangle$. Prove that the k-th column of the matrix $\widetilde{U}$ of (a) contains the coefficients of $u_k(x)$ as in

$$\widetilde{U} = \begin{bmatrix} 1 & u_{0,1} & u_{0,2} & u_{0,3} & \cdots & \cdots & u_{0,n} \\ 0 & 1 & u_{1,2} & u_{1,3} & \cdots & \cdots & u_{1,n} \\ 0 & 0 & 1 & u_{2,3} & \cdots & \cdots & u_{2,n} \\ \vdots & & 0 & 1 & \cdots & \cdots & u_{3,n} \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 & u_{n-1,n} \\ 0 & & & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(c) Derive a algorithm to compute the columns of $\widetilde{U}$ based on the formula (deduce it) that relates the k-th column u_k of U to its two "predecessors" u_{k-2}, u_{k-1} $(k = 3, \ldots, n)$.

(d) Prove that the algorithm of (c) uses $O(n^2)$ arithmetic operations.