Real Analysis Qualifying Exam
January 2009

Answer 4 out the following 5 questions. Always justify your answers.

1. Given $r > 1$ and $f, g \in L_r(S, \mathcal{S}, \mu)$, define the function

$$v(t) = \int_S |f + tg|^r d\mu, \quad t \in \mathbf{R}.$$

Prove that v is differentiable and find its derivative.

2. Let $(S, \mathcal{S}, \mu)$ be a measure space. a) Prove that, if $\mu(S) < \infty$, and each f_n, $n \in \mathbf{N}$, is measurable, then

$$f_n \to 0 \text{ in measure} \quad \Longleftrightarrow \quad \int_S \frac{|f_n|}{1 + |f_n|} d\mu \to 0.$$

b) Prove or disprove (e.g. by giving a counterexample) each of the two implications if $\mu(S) = \infty$.

3. a) Let f be the Cantor-Lebesgue function on $[0, 1]$ (f is defined following question 5). a1) Is f uniformly continuous?, a2) Is f of bounded variation? a3) Is f absolutely continuous?

b) Show that if g is absolutely continuous on $[a, b]$ then g transforms sets of (Lebesgue) measure zero into sets of measure zero (i.e., if $m(E) = 0$, then $m(g(E)) = 0$).

4. Suppose f is Borel measurable. Prove:

a) $\int_A f(x)dx = 0$ for all Borel sets A implies $f = 0$ a.e.

b) $\int_a^b f(x)dx = 0$ for all $-\infty < a < b < \infty$ implies $f = 0$ a.e.

c) $\int_a^b f(x)dx = 0$ for all $-\infty < a < b < \infty$, a, b rational, implies $f = 0$ a.e.

5. Justify, using $\dfrac{1}{u} = \displaystyle\int_0^\infty e^{-ux}dx$ (for $u > 0$) and real analysis theorems (but not complex analysis theorems), that

$$\lim_{b\to\infty} \int_0^b \frac{\sin u}{u} du = \frac{\pi}{2}.$$

Note: $\displaystyle\int e^{au} \sin u \; du = \frac{(a \sin u - \cos u)e^{au}}{1 + a^2} + C.$

[The Cantor-Lebesgue function: Let $x = \sum_{k=1}^\infty a_k 3^{-k}$ be the ternary expansion of $x \in [0, 1]$, with the convention that if $x = n3^{-k}$, n not a multiple of 3, then we take the expansion for which a_k is not 1. If each coefficient a_k of x is either 0 or 2, then $f(x) = \sum_{k=1}^\infty (a_k/2)2^{-k}$, and if some a_k are 1, then, if j_1 denotes the first index k for which $a_k = 1$, we take $f(x) = f(y)$ where $y = \sum_{k=1}^{j_1-1} a_k 3^{-k} + 2 \cdot 3^{-j_1}$.]