1. Prove or give a counter example: If X and Y are path connected then the product space $X \times Y$ is path connected.

2. Let $p : \mathcal{R} \longrightarrow S^1$ be the quotient map defined by $p(t) = e^{2\pi it}$. Prove the following two lifting properties:

   (i) **Unique Lifting Property**. Suppose B is connected, $\phi : B \longrightarrow S^1$ is continuous, and $\tilde{\phi}_1, \tilde{\phi}_2 : B \longrightarrow \mathcal{R}$ are lifts of ϕ that agree at some point of B. Then $\tilde{\phi}_1 = \tilde{\phi}_2$.

   (ii) **Path Lifting Property**. $f : [0,1] \longrightarrow S^1$ is any path, and $r_0 \in \mathcal{R}$ is any point in the fiber of p over $f(0)$. Then there exists a unique lift $\tilde{f} : [0,1] \longrightarrow \mathcal{R}$ of f such that $\tilde{f}(0) = r_0$.

3. Define an equivalence relation $\sim$ on $\mathcal{R}^2 - \{\vec{0}\}$ by declaring $\vec{x} \sim \vec{y}$ if $\vec{x}$ and $\vec{y}$ lie on the same straight line passing through the origin $\vec{0}$. Let $X = \mathcal{R}^2 - \{\vec{0}\}/\sim$ be the quotient space (i.e. identification space) determined by the equivalence relation $\sim$.

   Now define an equivalence relation $\approx$ on S^2 by declaring $\vec{x} \approx \vec{y}$ if $\vec{x} = -\vec{y}$. Let $Y = S^2/\approx$ be the quotient space (i.e. identification space) determined by the equivalence relation $\approx$.

   Prove or disprove: X is homeomorphic to Y.

4. Let X be a compact Hausdorff space and suppose $\{A_\alpha | \alpha \in J\}$ is a family of closed, connected subsets of X simply ordered by inclusion. Prove that $\cap_{\alpha \in J} A_\alpha$ is compact and connected.

5. If X is a connected space, a *cut point* of X is a point $x \in X$ such that $X - \{x\}$ is disconnected. For example, $\frac{1}{2}$ is a cut point of [0,1] but 0 is not. Let x be a cut point of a connected Hausdorff space X. If $\{U, V\}$ is a separation of $X - \{x\}$, that is $X - \{x\} = U \cup V$sep , then prove that $U \cup \{x\}$ is connected.

6. Let X be a first countable space. Prove that the following two properties are equivalent:

   (a) All compact subsets of X are closed sets.

   (b) X is Hausdorff.