Abstract Algebra Prelim Jan. 2011

1. Let R be a commutative ring with identity, and let M be an R-module. Recall the annihilator of M is $\mathrm{Ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$. For any ideal I in R, show M is an R/I-module by the rule $(r + I) \cdot m = rm$ if and only if $I \subseteq \mathrm{Ann}(M)$.

2. Let R be a commutative ring with identity, and let I and J be ideals in R. Recall that $I + J = \{r + r' \mid r \in I, r' \in J\}$, and IJ is the ideal generated by all products rr' with $r \in I$ and $r' \in J$.

   (a) Prove that if $I + J = R$ then $IJ = I \cap J$.

   (b) Assuming that $I + J = R$, show that for any a and b in R there exists some $x \in R$ such that $x \equiv a \bmod I$ and $x \equiv b \bmod J$. (Recall that $x \equiv a \bmod I$ if and only if $x - a \in I$.)

3. Let $\varphi\colon \mathbf{Z} \to \mathrm{Aut}(\mathbf{Z})$ by $n \mapsto \varphi_n$, where $\varphi_n(a) = (-1)^n a$. Define the semi-direct product group $G = \mathbf{Z} \rtimes_\varphi \mathbf{Z}$.

   (a) Write down the group law and the formula for inverses in G.

   (b) Find the center of G.

4. In a commutative ring R, an ideal Q is called *primary* if whenever any a and b in R satisfy $ab \in Q$ and $a \notin Q$, we have $b^n \in Q$ for some integer $n \geq 1$. (Equivalently, if $ab \equiv 0 \bmod Q$ and $a \not\equiv 0 \bmod Q$, we have $b^n \equiv 0 \bmod Q$ for some integer $n \geq 1$. That is, in the ring R/Q any zero divisor is nilpotent.) Show that the nonzero primary ideals in a PID are the ideals of the form (p^n) where p is a prime element and n is a positive integer. You may use that a PID is a UFD.

5. In $\mathbf{R}^3$ a *line-plane pair* is a pair of subspaces (V_1, V_2) where $V_1 \subset V_2$, $\dim V_1 = 1$, and $\dim V_2 = 2$. The *standard line-plane pair* in $\mathbf{R}^3$ is $(\mathbf{R}e_1, \mathbf{R}e_1 + \mathbf{R}e_2)$ where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Let $\mathcal{S}$ be the set of all line-plane pairs in $\mathbf{R}^3$.

   (a) The group $\mathrm{GL}(3, \mathbf{R})$ of invertible 3×3 real matrices acts on $\mathcal{S}$ by

   $$A \cdot (V_1, V_2) = (A(V_1), A(V_2)),$$

   where $A \in \mathrm{GL}(3, \mathbf{R})$ and $(V_1, V_2) \in \mathcal{S}$. Prove that the stabilizer subgroup of the standard line-plane pair is the group of invertible upper-triangular matrices in $\mathrm{GL}(3, \mathbf{R})$ (with arbitrary non-zero entries on the diagonal).

   (b) Prove that the $\mathrm{GL}(3, \mathbf{R})$-action on $\mathcal{S}$ is transitive.

6. Give examples as requested, with brief justification.

   (a) A maximal ideal in $\mathbf{C}[x, y]$ which contains the ideal $(xy, x^2 - 1)$.

   (b) A ring R and ideals I and J in R such that $IJ \neq I \cap J$.

   (c) A generator of the group of characters of $(\mathbf{Z}/7\mathbf{Z})^\times$.

   (d) A finite nonzero $\mathbf{Z}[i]$-module.