MATH 5410, Preliminary Exam

Department of Mathematics
University of Connecticut

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NAME: ______________________________ SIGNATURE: ____________________________

1. a) What is the definition of a self-adjoint operator from a Hilbert space H to itself;
b) Give an example of a self-adjoint operator for $H = L^2([0,1])$ and explain;
c) Prove that eigenvalues of a self-adjoint operator must be real.

2. a) What is the definition of weak convergence of a sequence $\{x_n\}$ in a Hilbert space H;
b) Prove that a strongly convergent sequence is also a weakly convergent sequence in H;
c) Give an example of a weakly convergent sequence which is NOT strongly convergent in l^2 and explain;

3. a) Give the definition of the limit of a sequence of distributions $\{f_n\}_1^\infty$ in R as $n \to \infty$.
b) Let

$$f(x) = e^{-x^2}, \ f_n(x) = nf(nx), \quad \forall x \in R, \quad n = 1, 2, \cdots.$$

How do you interpret function $f_n(x)$ as a distribution f_n in R?
c) Find the limit of $\{f_n\}_1^\infty$ as a sequence of distributions as $n \to \infty$.
(You may use the fact that $\int_R e^{-x^2} dx = \sqrt{\pi}$).

4. a) Suppose f is an operator from Banach space X to itself. Give the definition of f being Fréchet differentiable at a point $x \in X$.
b) Let $X = C[0,1]$ with sup-norm. Let $t_i \in [0,1]$ and $v_i \in C[0,1]$, and define $f(x) = \Sigma_{i=1}^n (x(t_i)^2) v_i$. Prove that f is Fréchet differentiable at all points of X and find a formula for f'.

5. Find a function in $C^1[0,1]$ that minimizes the integral $\int_0^1 [(u'(t))^2 + u^2(t)]dt$ with constraints $u(0) = 0$ and $u'(1) = 1$.

6. Let $[u_n]$ be an orthonormal sequence in a Hilbert space and let $[\lambda_n]$ be a bounded sequence in R. Prove that the operator $Ax = \Sigma \lambda_n < x, u_n > u_n$ is compact if and only if $\lambda_n \to 0$ as $n \to \infty$.