MATH 5510(313)   January 2011   PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer three out of six questions
You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

(a) Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Chebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, ...$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is a monic polynomial (that is, the leading coefficient is 1).

(b) Derive the formula for all the zeros of $T_n(x)$.

(c) Derive the formula for all the extrema of $T_n(x)$ in the closed interval $[-1, 1]$.

(d) Prove that $\hat{T}_n(x)$ has minimal infinity norm among all monic polynomials of degree n on the interval $[-1, 1]$. Moreover, show that $\|\hat{T}_n(x)\|_\infty = 1/2^{n-1}$, where $\| \cdot \|_\infty$ denotes the maximum norm of a function on the interval $[-1, 1]$.

(e) Prove that Chebyshev polynomials are orthogonal with respect to the inner product in Π_n defined by

$$< a(x), b(x) >= \int_{-1}^{1} \frac{a(x)b(x)}{\sqrt{1-x^2}} dx.$$

Q2) Answer 3 out of 3 questions (a), (b), (c).

(a) Prove that the Householder reflection matrix $P = I - 2ww^*$ (with $w^*w = 1$) is unitary and that $P^2 = I$.

(b) For a given vector x explain how to find w such that

$$Px = ke_1$$

with some k. Derive explicit formulas for w and k.

(c) Describe how, for a real matrix A, a sequence of Housholder reflections can be used to compute the QR factorization $A = QR$ with orthogonal Q and upper triangular R.

Q3) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

Derive a fast $O(n \log n)$ FFT-based algorithm for the polynomial multiplication problem, that is, given coefficients of two polynomials $a(x), b(x)$, compute the coefficients of their product $c(x) = a(x)b(x)$.

(a) Prove that the above polynomial multiplication problem is equivalent to the problem of multiplying a lower triangular Toeplitz matrix by a vector.

(b) Show how to "embed" a Toeplitz matrix into a circulant matrix, and justify the fact that the problem of (a) (that is, of multiplying a lower triangular Toeplitz matrix by a vector) can be solved via multiplying a circulant matrix by a vector.

(c) Prove that any circulant matrix C admits a factorization

$$C = FDF^*$$

where F is the DFT matrix and D is a diagonal matrix.

(d) Deduce the formula for the diagonal entries of D.

(e) Describe "in words" how the results of (a), (b), (c), and (d) allow us to compute the coefficients of $c(x) = a(x)b(x)$ in $O(n \log n)$ arithmetic operations.

Q4) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

(a) Prove that a positive definite matrix (partitioned as follows:)

$$A = \begin{bmatrix} d_1 & a_{21}^* \\ a_{21} & A_{22} \end{bmatrix}$$

admits a factorization

$$A = \begin{bmatrix} 1 & 0 \\ \frac{1}{d_1}a_{21} & I \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{d_1}a_{21}^* \\ 0 & I \end{bmatrix}$$

with some S, and deduce the formula for S.

(b) Prove that S is also positive definite.

(c) Use the results of (a) and (b) to prove that a positive matrix A admits a factorization

$$A = LDL^*,$$

where L is unit lower triangular (i.e., with 1's on the main diagonal), and D is a diagonal matrix with positive diagonal entries.

(d) Use the result of (c) to prove that a positive matrix A is always invertible and that its inverse is also a positive definite matrix.

(e) Use the result of (c) to prove that all the determinants of leading $k \times k$ submatrices of A are positive ($k = 1, 2, \ldots, n$).

Q5) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Let $\|x\|$ denotes the usual Euclidean norm $\sqrt{x^T x}$. Prove that the linear least squares problem

$$\min_{x \in \mathbb{R}^n} \|y - Ax\|$$

with a $m \times n$ matrix A has at least one minimal point x_0.

(b) Prove that if x_1 is another minimum point, then $Ax_0 = Ax_1$. The residual $r := y - Ax$ is uniquely determined and satisfies the equation $A^T r = 0$.

(c) Prove that Every minimum point x_0 is also a solution of normal equations

$$A^T Ax = A^T y$$

and conversely.

(d) Explain how the orthogonalization technique (that is, computing for the $m \times n$ matrix A the factorization $A = QR$ with $m \times m$ orthogonal matrix Q and $m \times n$ upper triangular matrix R) yields an efficient algorithm for solving the above least squares problem.

Q6) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Let T be an $n \times n$ positive definite matrix. Relate the factorization

$$T\widetilde{U} = \widetilde{L} \tag{1}$$

to the standard LDL^* factorization of T to prove that (1) always exists and it is unique. Here $\widetilde{U}$ is a unit (i.e., with 1's on the main diagonal) upper triangular matrix, and $\widetilde{L}$ is a lower triangular matrix.

(b) Let $\langle \cdot, \cdot \rangle$ be an arbitrary inner product in the vector space Π_n (of all polynomials whose degree does not exceed n). Let T be a positive definite moment matrix, i.e., $T = [\langle x^i, x^j \rangle]_{i,j=0}^n$. Let

$$u_k(x) = u_{0,k} + u_{1,k}x + u_{2,k}x^2 + \ldots + u_{k-1,k}x^{k-1} + x^k. \tag{2}$$

be the k-th orthogonal polynomial with respect to $\langle \cdot, \cdot \rangle$. Prove that the k-th column of the matrix $\widetilde{U}$ of (a) contains the coefficients of $u_k(x)$ as in

$$\widetilde{U} = \begin{bmatrix} 1 & u_{0,1} & u_{0,2} & u_{0,3} & \cdots & \cdots & u_{0,n} \\ 0 & 1 & u_{1,2} & u_{1,3} & \cdots & \cdots & u_{1,n} \\ 0 & 0 & 1 & u_{2,3} & \cdots & \cdots & u_{2,n} \\ \vdots & & 0 & 1 & \cdots & \cdots & u_{3,n} \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 & u_{n-1,n} \\ 0 & & & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(c) Assuming now that the moment matrix T has Toeplitz structure derive the so-called Levinson algorithm, that is, an algorithm to compute the columns of $\widetilde{U}$ based on the formula (deduce it) that relates the k-th column u_k of U to its "predecessor" u_{k-1} $(k = 2, 3, \ldots, n)$.

Hint: Use the fact (no need to prove it) that Toeplitz moment matrices T have the following property: if

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}$$

then

$$T \begin{bmatrix} x_n^* \\ x_{n-1}^* \\ x_{n-2}^* \\ \vdots \\ x_3^* \\ x_2^* \\ x_1^* \end{bmatrix} = \begin{bmatrix} y_n^* \\ y_{n-1}^* \\ y_{n-2}^* \\ \vdots \\ y_3^* \\ y_2^* \\ y_1^* \end{bmatrix}$$

(d) Prove that the algorithm of (c) uses $O(n^2)$ arithmetic operations.