Math 5410 Preliminary Exam

Jan 2012

Name Signature____

Do all 5 problems.

1. (a) Find the Green's Function $G(x,y)$ for operator A where
$$Ay = y'' + y$$
with $y'(0) = y(1) = 0$.

   (b) Define $T : L^2(0,1) \to L^2(0,1)$ such that for any $f \in L^2(0,1)$
$$Tf(x) = \int_0^1 G(x,y) f(y)\, dy.$$
Explain what spectral theorem is and why it is applicable.

   (c) Show that $\|T\| = \max\{|\lambda| : \ \lambda \text{ is an eigenvalue of } T\}$.

   (d) Compute $\|T\|$.

2. Let T be a compact operator on a Hilbert space H and $\{\varphi_n : \ n \in N\}$ be an orthonormal system of H.

   (a) Show $\varphi_n \rightharpoonup 0$ weakly. Explain why this gives an example of weakly convergent sequence which is not strongly convergent.

   (b) Using part (a). or otherwise, show $\|T\varphi_n\| \longrightarrow 0$

3. (a) Let λ_n be a sequence of complex numbers. Then operator S defined by $Sf = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi_n$ is compact iff $\lim_{n \longrightarrow \infty} \lambda_n = 0$.

   (a) Let f be an operator on a Banach space X, give the definition of f being Fréchet differentiable at a point $x \in X$.

   (b) Define $f : C[0,1] \longrightarrow C[0,1]$ by $[f(x)](t) = x(t) + \int_0^1 (x(st))^2\, ds$. Compute $f'(x)$.

4. Let $f(x) = e^{-x^2}$, $f_n(x) = nf(nx)$, $\forall x \in R$, $n = 1, 2 \cdots$.

   (a) Given the definition of the limit of a sequence of distributions in R.

   (b) Find the limit of $\{f_n\}_1^{\infty}$ as a sequence of distributions. You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

5. (a) Give the definition of a compact linear operator from a Banach space X to itself.

   (b) Given $X = L^2([0,1])$, find an example of compact linear opeator on X and explain why

   (c) Given $X = L^2([0,1])$, find an example of NON compact linear opeator on X and explain why.