Preliminary Exam in Complex Analysis
January 2012

Instructions

All assertions require written justification. In particular, state and verify the hypotheses of any theorems you use. In complex analysis the terms 'analytic' and 'holomorphic' are used interchangeably.

You are free to make use of the following without justification: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

1. In the following three part problem, either prove the assertion or give a counterexample.

(a) Suppose Ω is a simply connected region, f is analytic in Ω and $f'(z) \neq 0$ for all $z \in \Omega$. Then f is one-to-one in Ω.

(b) If f is a non-constant, entire function that satisfies $|f(z)| \leq 2|ze^z|$, then f has an essential singularity at infinity.

(c) Let f be an analytic function in the punctured disk, $\Delta_0 = \{z \,|\, 0 < |z| < 1\}$, with a pole at 0. Then there is a value $M > 0$, so that for any w with $|w| > M$ there is a $z \in \Delta_0$ for which $f(z) = w$.

2. Find the Laurent expansions of $f(z) = \dfrac{z+2}{z^2 - z - 2}$ in powers of z and $1/z$, converging in the indicated domains

(a) $\{z \,|\, 1 < |z| < 2\}$

(b) $\{z \,|\, 2 < |z| < \infty\}$.

3. Let $s, R > 0$. By integrating $f(z) = e^{-z^2}$ over the rectangle with vertices $\pm R$, $\pm R + is$ and letting R approach ∞, find the Fourier transform $\displaystyle\int_{-\infty}^{\infty} e^{-ist} e^{-t^2}\,dt$.

4. Suppose that f is analytic in the open unit disk $\Delta = \{z \,|\, |z| < 1\}$, continuous on the closed unit disk and $|f(z)| = 1$ for $|z| = 1$. Show that

(a) f can be extended to be analytic in $\mathbb{C}$, except for finitely many poles, by defining

$$f(z) = \left(\overline{f\left(\frac{1}{\bar{z}}\right)}\right)^{-1}$$

in the exterior of the unit disk Δ.

(b) f is rational and has the form $f(z) = \lambda \dfrac{z - a_1}{1 - \bar{a}_1 z} \cdots \dfrac{z - a_n}{1 - \bar{a}_n z}$, $|a_i| < 1$ and $|\lambda| = 1$.