MATH 5510(313) January 2012 PRELIMINARY EXAMINATION

INSTRUCTIONS: Solve three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) Solve (a), (b), (c), (d)

(a) Consider the following classical interpolation problem.

> - **Given** $n+1$ support points
>
> $$(x_i, f_i) \qquad i = 0, \ldots, n; \qquad\qquad (x_i \neq x_j \quad for \quad i \neq j).$$
>
> - **Find** a polynomial $P(x)$ whose degree does not exceed n such that
>
> $$P(x_i) = f_i, \qquad i = 0, \ldots n.$$

Define the Vandermonde matrix, and then reformulate the above interpolation problem as a matrix problem of solving a linear system of equations with the Vandermonde coefficient matrix.
Use the condition

$$x_i \neq x_j \quad for \quad i \neq j,$$

to prove that the Vandermonde matrix is nonsingular.
Use the latter fact to prove that the classical interpolation problem stated above has a unique solution.

(b) Let $P_{i_0 i_1 \ldots i_k}(x)$ be the (unique) polynomial that interpolates at points

$$(x_{i_m}, f_{i_m}) \qquad m = 0, \ldots k.$$

Prove that there exists a unique coefficient $f_{i_0 \ldots i_k}$ such that

$$P_{i_0 \ldots i_k}(x) = P_{i_0 \ldots i_{k-1}} + f_{i_0 \ldots i_k}(x - x_{i_0})(x - x_{i_1}) \cdots (x - x_{i_{k-1}}).$$

(c) Prove the recursion:

$$f_{i_0 \ldots i_k} = \frac{f_{i_1 \ldots i_k} - f_{i_0 \ldots i_{k-1}}}{x_{i_k} - x_{i_0}}.$$

(d) Prove the following theorem (error in polynomial interpolation).
If the function f has an $(n+1)$st derivative, then for every argument $\bar{x}$ there exist a number ξ (in the smallest interval containing $x_{i_0}, x_{i_1}, \ldots x_{i_n}, \bar{x}$), satisfying

$$f(\bar{x}) - P_{i_0, i_1, \ldots, i_n}(x) = \frac{w(\bar{x}) f^{(n+1)}(\xi)}{(n+1)!},$$

where

$$w(x) = (x - x_{i_0})(x - x_{i_1}) \dots (x - x_{i_n}).$$

Q2) Solve (a), (b), (c)

(a) Use the fact that each norm $\|\cdot\|$ on $\mathbb{C}^n$ is uniformly continuous (no need to prove the latter fact, just formulate it as a specific inequality) to prove the following theorem.

All norms on $\mathbb{C}^n$ are equivalent in the following sense. For each pair of norms $p_1(x)$ and $p_2(x)$ there are positive constants m and M satisfying

$$mp_2(x) \le p_1(x) \le Mp_2(x)$$

for all x.

(b) Prove that if F is an $n \times n$ matrix with $\|F\| < 1$, then $(I+F)^{-1}$ exists and satisfies

$$\|(I+F)^{-1}\| \le \frac{1}{1 - \|F\|}.$$

(c) Let A be a nonsingular $n \times n$ matrix, $B = A(I+F)$, $\|F\| < 1$, and x and Δx be defined by

$$Ax = b, \qquad B(x + \Delta x) = b.$$

Use **(b)** to prove that

$$\frac{\|\Delta x\|}{\|x\|} \le \frac{\|F\|}{1 - \|F\|}$$

as well as

$$\frac{\|\Delta x\|}{\|x\|} \le \frac{cond(A)}{1 - cond(A)\frac{\|B-A\|}{\|A\|}} \cdot \frac{\|B-A\|}{\|A\|}$$

if

$$cond(A)\frac{\|B-A\|}{\|A\|} < 1.$$

Q3) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Define a Hankel matrix. Let H be an $n \times n$ positive definite Hankel matrix. Relate the factorization

$$H\widetilde{U} = \widetilde{L} \tag{1}$$

to the standard LDL^* factorization of H to prove that (1) always exists and it is unique. Here $\widetilde{U}$ is a unit (i.e., with 1's on the main diagonal) upper triangular matrix, and $\widetilde{L}$ is a lower triangular matrix.

(b) Let $\langle \cdot, \cdot \rangle$ be an inner product in the vector space Π_n (of all polynomials whose degree does not exceed n). Let the above Hankel matrix H be a moment matrix, i.e., $H = [\langle x^i, x^j \rangle]_{i,j=0}^n$. Let

$$u_k(x) = u_{0,k} + u_{1,k}x + u_{2,k}x^2 + \dots + u_{k-1,k}x^{k-1} + x^k. \tag{2}$$

be the k-th orthogonal polynomial with respect to $\langle \cdot, \cdot \rangle$. Prove that the k-th column of the matrix $\widetilde{U}$ of (a) contains the coefficients of $u_k(x)$ as in

$$\widetilde{U} = \begin{bmatrix} 1 & u_{0,1} & u_{0,2} & u_{0,3} & \cdots & \cdots & u_{0,n} \\ 0 & 1 & u_{1,2} & u_{1,3} & \cdots & \cdots & u_{1,n} \\ 0 & 0 & 1 & u_{2,3} & \cdots & \cdots & u_{2,n} \\ \vdots & & 0 & 1 & \cdots & \cdots & u_{3,n} \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 & u_{n-1,n} \\ 0 & & & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(c) Derive a algorithm to compute the columns of $\widetilde{U}$ based on the formula (deduce it) that relates the k-th column u_k of U to its two "predecessors" u_{k-2}, u_{k-1} $(k = 3, \ldots, n)$.

(d) Prove that the algorithm of (c) uses $O(n^2)$ arithmetic operations.

Q4) Solve (a), (b), (c)

(a) Prove: An $n \times n$ matrix $A = \left[\ a_{i,j}\ \right]$ admits an LU-factorization $A = LU$ without pivoting and with invertible factors L and U if and only if for $k = 1, \ldots n$ the leading principal submatrices of A of order k are all invertible.

(b) Let A be an $n \times n$ invertible matrix that admits an LU-factorization without pivoting. Show that such a factorization is unique; namely, if $A = L_1U_1 = L_2U_2$, where L_1 and L_2 are lower triangular matrices with $diag(L_1) = diag(L_2) = I$ and where U_1 and U_2 are upper triangular, then $L_1 = L_2$ and $U_1 = U_2$.

(c) Suppose that A is a real $n \times n$ symmetric invertible matrix which admits an LU-factorization $A = LU$, with a lower triangular matrix L such that $diag(L) = I$, and with an upper triangular matrix U having positive diagonal entries. Show that A admits a factorization $A = LL^T$.