Measure and Integration Prelim, January 2012

1. Let μ be a finite Borel measure on $[0,1]$. Suppose that $f : [0,1]^2 \to \mathbb{R}_+$ is a **non-negative** measurable function on $[0,1]^2$ such that for each fixed x, the function $u \to f(x,u)$ is continuous on $[0,1]$.

$$I_\gamma(f) := \int_{[0,\gamma]} \int_{[0,1]} \left(e^{-\int_0^u (1+f(x,v/\gamma))dv} \right) d\mu(x) \times du$$

   Find $\lim_{\gamma\to\infty} I_\gamma(f)$.

2. Let G be a right-continuous nondecreasing function on $\mathbb{R}$. Assuming without proof that G is differentiable a.e. with respect to Lebesgue measure, find two proofs to

$$\int_{[a,b]} G'(x)dx \le G(b) - G(a), \quad -\infty < a < b < \infty$$

   (a) One proof based on results on differentiation of measures.

   (b) One direct proof. (Hint : consider $g_n(x) = n(G(x+\frac{1}{n}) - G(x))$ over $[a,b]$).

3. Let $f \in L^1(\mathbb{R})$ and let $p \in (1,\infty)$. Prove that $f \in L^p(\mathbb{R})$ if and only if there exists a bounded continuous nondecreasing function $G : \mathbb{R} \to \mathbb{R}$ such that

$$\left| \frac{\int_{[a,b]} f(x)dx}{b-a} \right|^p \le \frac{G(b) - G(a)}{b-a}.$$

   (Hint. One direction requires Lebesgue's differentiation theorem and the result of Problem 2. The other direction follows from an inequality on L^p-spaces).

4. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f, f_1, \ldots$ be a sequence of real-valued $\mathcal{F}$-measurable functions on X.

   (a) Define what it means to say $\lim_{n\to\infty} f_n = f$ in μ-measure.

   (b) Suppose that μ is a finite measure. Show that if $\lim_{n\to\infty} f_n = 0$ in μ- measure and $\limsup_{n\to\infty} \int |f_n|^2 d\mu < 1$, then $\lim_{n\to\infty} f_n = 0$ in $L^r(\mu)$ for every $r \in [1,2)$.

   (c) Show by counterexample that :

      i. The conclusion of (b) may not hold in the case $r = 2$.

      ii. The conclusion of (b) may not hold if μ is not a finite measure.

5. (a) Let $f : \mathbb{R} \to \mathbb{R}$. Define what it means to say that f is absolutely continuous.

   (b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous and $N \subset \mathbb{R}$ is a Lebesgue null set (a Lebesgue measurable set with Lebesgue measure zero), then $f(N)$ is a Lebesgue null set.