Abstract Algebra Prelim Jan. 2013

1. In the group $\mathrm{Aff}(\mathbf{Z}/(7)) = \{(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}) : a, b \in \mathbf{Z}/(7), a \neq 0\}$, compute the number of p-Sylow subgroups for each prime p dividing the order of the group.

2. Let $D_n = \langle r, s \rangle$ be the nth dihedral group for $n \geq 3$ (order $2n$, $r^n = 1, s^2 = 1, sr = r^{-1}s$).

   (a) For any automorphism f of D_n, show $f(r) = r^a$ for some integer a such that $(a, n) = 1$, and $f(s) = r^b s$ for some integer b.

   (b) Conversely, given integers a and b such that $(a, n) = 1$, show there is a unique automorphism f of D_n such that $f(r) = r^a$ and $f(s) = r^b s$.

3. Let F be an infinite field.

   (a) If $f(X) \in F[X]$ satisfies $f(a) = 0$ for all $a \in F$, then prove $f(X) = 0$ in $F[X]$.

   (b) If $f(X, Y) \in F[X, Y]$ satisfies $f(a, b) = 0$ for all $(a, b) \in F \times F$, then prove $f(X, Y) = 0$ in $F[X, Y]$.

4. (a) If a commutative ring R has exactly one maximal ideal, then prove this ideal must be $R - R^\times$ (the complement of the units in R).

   (b) Let R be the ring of rational numbers with an odd denominator: $R = \{a/b : a, b \in \mathbf{Z}, b \text{ odd}\}$. Describe $R^\times$ and show R has a unique maximal ideal.

5. Let A be a commutative ring with identity. For an ideal I in A and $a \in A$ define

$$(I : a) = \{c \in A : ca \subset I\}.$$

   (a) Show $(I : a)$ is an ideal in A and it contains I.

   (b) If the ideals $I + Aa$ and $(I : a)$ are both finitely generated then show I is finitely generated. More precisely, if $I + Aa$ is generated by $x_1 + b_1 a, \ldots, x_m + b_m a$ ($x_i \in I$, $b_i \in A$) and $(I : a)$ is generated by $y_1, \ldots, y_n$, then show I is generated by $x_1, \ldots, x_m, y_1 a, \ldots, y_n a$.

   (c) Assume A contains an ideal that is not finitely generated. Prove A contains a prime ideal that is not finitely generated. (Hint: Use Zorn's lemma to show there is an ideal P in A that is not finitely generated and contained in no other ideal that is not finitely generated. Then use part b to show P is prime.)

6. Give examples as requested, with brief justification.

   (a) A subgroup of $\mathbf{Z} \times \mathbf{Z}$ that is not equal to $a\mathbf{Z} \times b\mathbf{Z}$ for integers a and b.

   (b) A group isomorphism from $\mathbf{Z}/6\mathbf{Z}$ to $(\mathbf{Z}/7\mathbf{Z})^\times$.

   (c) A ring isomorphism from $\mathbf{R}[x]/(x^4 - 2)$ to $\mathbf{R} \times \mathbf{R} \times \mathbf{C}$.

   (d) A unit in $\mathbf{Z}[x]/(x^3)$ other than ± 1.