Math 5410 Preliminary Exam

Jan 2013

Name Signature____________________________________

Do all 5 problems.

1. (a) Find the Green's Function $G(x, y)$ for operator A where

$$Ay = y'' + y$$

   with $y'(0) = y(1) = 0$.

   (b) Find the operator norm of the Green's operator from L^2 to L^2.

2. Let T be a bounded operator on a Hilbert space H and $\{\varphi_n : \ n \in N\}$ be an orthonormal system of H.

   (a) Show $\varphi_n \rightharpoonup 0$ weakly.

   (b) Using part (a) or otherwise, show that if T is compact, then $\|T\varphi_n\| \longrightarrow 0$.

   (c) If $\sum_{n=1}^{\infty} \|T\varphi_n\|^2 < \infty$, then T is compact.

3. Find $\left(\Delta - k^2\right)\left(\frac{1}{4\pi r} e^{-kr}\right)$ in the sense of distributional derivatives. Here $r = \sqrt{x^2 + y^2 + z^2}$.

4. Let $f(x) = \frac{1}{1+x^2}$, $f_n(x) = nf(nx)$, $\forall x \in R$, $n = 1, 2 \cdots$.

   (a) Give the definition of the limit of a sequence of distributions on $C_0^{\infty}(R)$.

   (b) Find the limit of $\{f_n\}_1^{\infty}$ as a sequence of distributions.

5. Let H be a Hilbert space and T be a bounded linear operator on H.

   (a) Give the definition of the adjoint operator T^*. Show that T^* is well defined and bounded.

   (b) Given $H = L^2([0, 1])$, find an example of self-adjoint operator on H.

   (c) Show that eigenvalues of self-adjoint operators must be real.