Real Analysis Qualifying Exam, Winter 2013

Below m denotes the Lebesgue measure on the Lebesgue σ-algebra on the real line $\mathbb{R}$, and $(X, \mathcal{A}, \mu)$ denotes any sigma-finite measure space.

1. a) Give an example of a sequence of functions that converges to zero in $L^1([0,1], m)$ but not almost everywhere (a.e.).

b) Suppose that f_n, $n \in \mathbb{N}$, are in $L^1([0,1], m)$ and there exists $\delta > 0$ such that $\|f_n\|_{L^1} \leq n^{-1-\delta}$. Prove that f_n converges to zero a.e.

2. a) Suppose $f \in L^1(X, \mathcal{A}, \mu)$. Prove that if $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_A |f| \, d\mu < \epsilon$$

whenever $A \in \mathcal{A}$ with $\mu(A) < \delta$.

b) Suppose $f_k \to f$ in $L^1(X, \mathcal{A}, \mu)$. Prove that if $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_A |f_k| \, d\mu < \epsilon$$

whenever $k \geq 1$ and $A \in \mathcal{A}$ with $\mu(A) < \delta$.

3. a) Prove the generalized Minkowski inequality, that is, prove that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are sigma–finite measure spaces and $f : X \times Y \mapsto \mathbb{R}$ is $\mathcal{A} \otimes \mathcal{B}$–measurable, then

$$\Big\| \|f\|_{L^1(X,\mu)} \Big\|_{L^p(Y,\nu)} \leq \Big\| \|f\|_{L^p(Y,\nu)} \Big\|_{L^1(X,\mu)}$$

for all $p > 1$.

Hint: show that if $f \geqslant 0$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\|g\|_{L^q(Y,\mathcal{B},\nu)} = 1$, *then*

$$\int_X \left(\int_Y f(x,y) g(y) d\nu(y) \right) d\mu(x) \leq \Big\| \|f(x,\cdot)\|_{L^p(Y,\mathcal{B},\nu)} \Big\|_{L^1(X,\mathcal{A},\mu)} \quad \textit{and}$$

$$\sup_g \int_Y \left(\int_X f(x,y) d\mu(x) \right) g(y) d\nu(y) = \Big\| \|f(\cdot,y)\|_{L^1(X,\mathcal{A},\mu)} \Big\|_{L^p(Y,\mathcal{B},\nu)}.$$

b) Assuming the generalized Minkowski inequality, show that if $p > 1$ and $f \in L^p([0,\infty), m)$, then the 'mean functional' of f,

$$F(y) := \frac{1}{y} \int_0^y f(t) dt = \int_0^1 f(xy) dx,$$

is also in $L^p([0,\infty), m)$ and moreover

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p$$

where $\|\cdot\|_p$ stands for the $L^p([0,\infty), m)$–norm.

Hint: consider $f(xy)$ as a function of two variables on $[0,\infty) \times [0,\infty)$.

4. Let f be the Cantor function, that is: for x in the Cantor set $C \subset [0,1]$, $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \in \{0,2\}$ for all $k \in \mathbb{N}$, f is defined as $f(x) = \sum_{k=1}^{\infty}(a_k/2)2^{-k}$, f is constant on the intervals removed from $[0,1]$ to form C, and f is continuous.

Make a sketch of function f and answer the following questions (provide brief explanations but not necessarily complete proofs):

a) Is f uniformly continuous?

b) Is f of bounded variation?

c) Is f absolutely continuous?

d) Define a set function ν on $[0,1)$ by $\nu[a,b) = f(b) - f(a)$; does ν extend to a Borel measure, and if so, what is the Lebesgue decomposition of ν?