Name: ______________________________

Math 5410 Prelim January, 17, 2014

(1a) State and prove an existence and uniqueness theorem for the equation $\frac{d^2x}{dt^2} + f(x) = 0$ with initial conditions $x(0) = a$ and $x'(0) = b$ under the assumption that f and its partial derivatives are continuous. (You can assume the Contraction Mapping Theorem).
(1b) Let $a = b = f(0) = 0$ in part (a). Can $x(t) = t^3$ be a solution to part (a)? Explain.

(2a) Find the Green's function $G(x, y)$ for the operator A where

$$Au = u'' - 4u$$

with $u(0) = u'(1) = 0$.
If $Au = f(x)$, express the function u in terms of G and f.

(2b) Define $T : L^2(0, 1) \to L^2(0, 1)$ such that for any $f \in L^2(0, 1)$,

$$(Tf)(x) = \int_0^1 G(x, y)f(y)\, dy \; .$$

Explain what the spectral theorem is and why it is applicable.

(2c) Show that $\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of T}\}$.
(2d) Compute $\|T\|$. (hint: find eigenvalues of A).

(3) Let

$$U(x, y) = 1/(x^2 + y^2 + z^2)$$

Compute distributionally $\Delta U = (\partial_x^2 + \partial_y^2 + \partial_z^2)U$ in $\Re^3$.

(4) Let H be a Hilbert space and $K : H \to H$ is a linear, bounded, compact operator. Define $A = I + K$. Show that if A is injective, then it is surjective.

5 Is the map $F : L^3(0, 1) \to R^1$ defined by $F(u) = \int_0^1 (u(x)^3 dx$ Frechet differentiable? (Justify your answer.) If yes, identify the derivative.