COMPLEX ANALYSIS PRELIMINARY EXAMINATION

January 2014

Instructions:

- Justify your answers, and show all your work.
- $\mathbb{C}$ **the space of complex numbers**;
- $B(a, R) = \{z \in \mathbb{C} : |z - a| < R\}$;
- $D = B(0, 1)$**, the unit disk in** $\mathbb{C}$;

1. Let G be a bounded connected open subset of $\mathbb{C}$, and let f be a nonconstant continuous function on $\overline{G}$ which is holomorphic on G. Assume that $|f(z)| = 1$ for all $z \in \partial G$. Show that f has at least one zero in G.

2. Let $f(z)$ be holomorphic in the right half-plane $H := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, with $|f(z)| < 1$ for all $z \in H$. If $f(1) = 0$, how large can $|f(2)|$ be?

3. Evaluate and **justify** your answer

$$\int_0^\infty \frac{x^2}{x^4 + x^2 + 1} dx.$$

Hint: you can use the identity $a^6 - 1 = (a^2 - 1)(a^4 + a^2 + 1)$.

4. Suppose f is entire, and

$$\int_0^{2\pi} |f(re^{it})| dt \leqslant r^{13/4}$$

for all $r > 0$. Prove that $f \equiv 0$.

5. Prove that there is no function f that is holomorphic in the punctured disk $D \backslash 0$, and f' has a simple pole at 0.

6. Let f be a complex-valued function in the unit disk D such that both functions f^2 and f^3 are holomorphic in D. Prove that f is holomorphic as well.