Qualifying Exam

1. Let $A \in \mathbb{R}^{n\times n}$. Show that if $Ax = b$ has at least one solution for any $b \in \mathbb{R}^n$, then $Ax = b$ has exactly one solution for any $b \in \mathbb{R}^n$.

2. Let $A \in \mathbb{R}^{n\times n}$ be any non-singular matrix and $\|\cdot\|$ be any vector norm. Show that the function $N(x) = \|Ax\|$ is a vector norm.

3. Let $A \in \mathbb{R}^{n\times m}$. Define $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$, and $\|A\|_F$ (one, two, infinity and the Frobenius norms.) Compute them in the case of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

4. (a) Suppose that $p(x)$ is a polynomial of degree at most n which has $n+1$ distinct roots. Show that $p(x) \equiv 0$. Use this result to show that the polynomial $p_n(x)$, of order at most n, which interpolates a function f at $n+1$ distinct points $x_0, \dots, x_n$ is unique. (Assume that the values which f takes at these points are $f_0, \dots, f_n$, respectively.)

   (b) Suppose that $f \in C^{n+1}[a, b]$ and that $x_0, \dots, x_n$ are $n+1$ distinct points in the interval. Let p_n be the interpolation polynomial for f on $x_0, \dots, x_n$. Let $e_n(x) = f(x) - p_n(x)$ denote the error function on $[a, b]$. Show that for each point $x \in [a, b]$, there is a point $\xi_x \in (a, b)$ such that

$$e_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n).$$

   (c) A function f is defined on the interval $[0, 1]$ and its derivatives satisfy that $|f^{(m)}(x)| \le m!$, for all $x \in [0, 1]$ and for all $m = 0, 1, \dots$. For any $0 < q < 1$, let $p_n(x)$, $n > 0$, be the interpolation polynomial of degree at most n which interpolates f at $x_0 = 1$, $x_1 = q$, $x_2 = q^2$, $\dots$, $x_n = q^n$. Show that

$$\lim_{n\to\infty} p_n(0) = f(0)$$

   Taking $q = 1/2$ and $n = 10$, find an upper estimate on $|p_{10}(0) - f(0)|$.

5. (a) Find $\{p_0, p_1, p_2\}$ such that p_i is a polynomial of degree i and this set is orthogonal on $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$. (Formulas $\int x^n e^{-x} dx = n!$ and $0! = 1$ may be useful.)

   (b) Derive the two-point Gaussian formula

$$\int_0^\infty f(x)e^{-x}dx \approx w_1 f(x_1) + w_2 f(x_2),$$

   i.e. find the weights and the nodes.

6. One wants to solve the equation $x + \ln x$, whose root is near 0.5, using one or more of the methods:

$$(i)\ x_{k+1} = -\ln x_k \qquad (ii)\ x_{k+1} = e^{-x_k} \qquad (iii)\ x_{k+1} = \frac{x_k + e^{-x_k}}{2}$$

   (a) Which of the three methods can be used?

   (b) Which method should be used?

   (c) Give better iterative formula.

   Justify your answers.