Measure and Integration Preliminary Exam, January 2014

Solve four problems. Indicate which four.

1. Let h be a bounded measurable function on $\mathbb{R}$ such that

$$\lim_{x\to\pm\infty} \frac{1}{x}\int_0^x h(t)dt = 0.$$

Show that for all functions $F \in L^1(\mathbb{R})$,

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} F(t)h(\lambda t)dt = 0.$$

Does $h(t) = \cos t$ satisfy the hypothesis, hence the conclusion? Hint: Start with indicators of intervals.

2. Prove that if $f : [0,\infty) \mapsto [0,\infty)$ is non-increasing and Lebesgue integrable, then $\lim_{x\to\infty} xf(x) = 0$. Give an example of a continuous Lebesgue integrable function on $[0,\infty)$ for which $\limsup_{x\to\infty} xf(x) = \infty$.

3. Let $(S, \mathcal{S}, \mu)$ be a measure space.

(a) Prove that, if $\mu(S) < \infty$, and each f_n, $n \in \mathbb{N}$, is measurable, then

$$f_n \to 0 \text{ in measure} \iff \int_S \frac{|f_n|}{1+|f_n|}d\mu \to 0.$$

(b) Prove or disprove (e.g. by giving a counterexample) each of the two implications if $\mu(S) = \infty$.

4. (a) Use a real analysis theorem to show that $\sum_k \sum_\ell a_{k\ell} = \sum_\ell \sum_k a_{k\ell}$ for $a_{k\ell} \geq 0$. Then show that if μ_k, $k \in \mathbf{N}$, are measures on $(S, \mathcal{S})$, so is the set function μ given by $\mu(A) = \sum_{k=1}^\infty \mu_k(A)$, $A \in \mathcal{S}$.

(b) Let $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{B})$, assume $\sum_k \mu_k[-n,n]$ is finite for all n, and let $\mu_k = \lambda_k + \nu_k$ be the Lebesgue decomposition of μ_k for each k (λ_k and Lebesgue measure m are mutually singular, and ν_k is absolutely continuous w.r.t. m). Prove that if $\lambda = \sum_k \lambda_k$ and $\nu = \sum_k \nu_k$, then $\mu = \lambda + \nu$ is the Lebesgue decomposition of μ.

(c) Show that if $F_k : [a,b] \mapsto [0,\infty)$ are non-decreasing, right continuous and non-negative functions, and if $F(x) := \sum_k F_k(x) < \infty$ for all x in $[a,b]$, then F is also right continuous (and, obviously, non-decreasing and non-negative) and

$$F'(x) = \sum_k F_k'(x)$$

for almost all x in $[a,b]$.

5. (a) Prove the generalized Minkowski inequality, that is, prove that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are sigma–finite measure spaces and $f : X \times Y \mapsto \mathbb{R}$ is $\mathcal{A} \otimes \mathcal{B}$–measurable, then

$$\left\| \|f\|_{L_1(X,\mu)} \right\|_{L_p(Y,\nu)} \leq \left\| \|f\|_{L_p(Y,\nu)} \right\|_{L_1(X,\mu)}$$

for all $p \geq 1$. Hint: duality of L_p spaces and a famous inequality may help.

(b) Let $\|\cdot\|_p$ stand for the $L_p(\mathbb{R})$–norm with respect to the Lebesgue measure. Show that if $p > 1$ and $f \in L_p(\mathbb{R})$, then the 'mean functional' of f,

$$F(x) := \frac{1}{x}\int_0^x f(y)dy = \int_0^1 f(xt)dt,$$

is also in $L_p(\mathbb{R})$ and, moreover,

$$\|F\|_p \leq q\|f\|_p$$

where q is conjugate of p, that is $p^{-1} + q^{-1} = 1$.