Abstract Algebra Prelim Jan. 2015

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. Let G be a group and Z be its center.

   (a) Prove that if G/Z is cyclic then G is abelian.

   (b) Prove that if G is a nontrivial finite p-group then Z is nontrivial.

   (c) Prove that if G is a nontrivial finite p-group and every abelian normal subgroup of G is contained in Z then G is abelian. (Hint: If G were not abelian, look at the center of G/Z. Ideas from parts a and b may be useful here.)

2. The prime factorization of 2015 is $5 \cdot 13 \cdot 31$. Prove every group of order 2015 has a cyclic normal subgroup of order $13 \cdot 31 = 403$, and explain why a non-abelian group of order 2015 exists.

3. Let R be a commutative ring with unity, and let I be a *proper* ideal of R.

   (a) Prove there is a one-to-one correspondence between the ideals of the quotient ring R/I and the ideals of R that contain I.

   (b) Under the correspondence in part a, prove prime ideals of R/I correspond to prime ideals of R that contain I.

4. Let R be an integral domain.

   (a) Let $a, b \in R$ and assume a is a unit. Prove the substitution homomorphism $\varphi\colon R[x] \to R[x]$ where $\varphi(f(x)) = f(ax + b)$ for all $f(x) \in R[x]$ is an automorphism of $R[x]$ and it fixes each element of R.

   (b) Prove a converse to part a: every ring automorphism of $R[x]$ that fixes each element of R is of the form described in part a.

5. Let R be a commutative ring with identity. An element $e \in R$ is called an *idempotent* if $e^2 = e$. Prove that if I and J are ideals in R such that $R = I \oplus J$ (that is, $R = I + J$ and $I \cap J = \{0\}$) then there is an idempotent e in R such that $I = Re$ and $J = R(1 - e)$. (Hint: If e is an idempotent, then $1 - e$ is also an idempotent.)

6. Give examples as requested, with brief justification.

   (a) An automorphism of the group $(\mathbf{Z}/11\mathbf{Z})^\times$ other than the identity and inversion.

   (b) A prime factorization of 15 in $\mathbf{Z}[i]$.

   (c) An explicit construction of a field of size 9.

   (d) An irreducible cubic polynomial in $\mathbf{F}_5[x]$.