MATH 5510 January 2015 PRELIMINARY EXAMINATION

INSTRUCTIONS: Solve three out of five questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) Solve (a), (b), (c), (d), (e).

(a) Define the $n \times n$ Vandermonde matrix V_n (with the nodes $x_1, x_2, \ldots, x_n$), and derive the factorization:

$$V_n = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & \vdots \\ 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & x_2 - x_1 & \ddots & & \vdots \\ \vdots & \ddots & x_3 - x_1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_n - x_1 \end{bmatrix}}_{L_1^{-1}} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_{n-1} \end{array}\right] \underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 0 & 1 & x_1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & x_1^2 \\ \vdots & & \ddots & \ddots & x_1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}}_{U_1^{-1}}$$

(b) Derive the formula for the determinant of V_n. Use the condition

$$x_i \neq x_j \quad for \quad i \neq j,$$

to prove that the Vandermonde matrix is nonsingular.

(c) Use (b) to prove that the following classical interpolation problem has a unique solution.

- **Given** n support points

$$(x_i, f_i) \qquad i = 1, \ldots, n; \qquad\qquad (x_i \neq x_j \quad for \quad i \neq j).$$

- **Find** a polynomial $P(x)$ whose degree does not exceed $(n-1)$ such that

$$P(x_i) = f_i, \qquad i = 1, \ldots n.$$

(d) Use (a) to recursively to derive the formula for factoring V_n^{-1} into a product of $n-1$ lower triangular matrices and $n-1$ upper triangular matrices. Use it to derive the Bjorck-Pereyra algorithm for solving the interpolation problem of (c).

(e) Prove that the Bjorck-Pereyra algorithm has the cost of $O(n^2)$ operations

Q2) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

(a) Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Chebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, ...$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is a monic polynomial (that is, the leading coefficient is 1).

(b) Derive the formula for all the zeros of $T_n(x)$.

(c) Derive the formula for all the extrema of $T_n(x)$ in the closed interval $[-1, 1]$.

(d) Prove that $\hat{T}_n(x)$ has minimal infinity norm among all monic polynomials of degree n on the interval $[-1, 1]$. Moreover, show that $\|\hat{T}_n(x)\|_\infty = 1/2^{n-1}$, where $\|\cdot\|_\infty$ denotes the maximum norm of a function on the interval $[-1, 1]$.

(e) Prove that Chebyshev polynomials are orthogonal with respect to the inner product in Π_n defined by

$$< a(x), b(x) >= \int_{-1}^{1} \frac{a(x)b(x)}{\sqrt{1-x^2}} dx.$$

Q3) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Let T be an $n \times n$ positive definite matrix. Relate the factorization

$$T\widetilde{U} = \widetilde{L} \tag{1}$$

to the standard LDL^* factorization of T to prove that (1) always exists and it is unique. Here $\widetilde{U}$ is a unit (i.e., with 1's on the main diagonal) upper triangular matrix, and $\widetilde{L}$ is a lower triangular matrix.

(b) Let $\langle \cdot, \cdot \rangle$ be an arbitrary inner product in the vector space Π_n (of all polynomials whose degree does not exceed n). Let T be a positive definite moment matrix, i.e., $T = [\langle x^i, x^j \rangle]_{i,j=0}^n$. Let

$$u_k(x) = u_{0,k} + u_{1,k}x + u_{2,k}x^2 + \ldots + u_{k-1,k}x^{k-1} + x^k. \tag{2}$$

be the k-th orthogonal polynomial with respect to $\langle \cdot, \cdot \rangle$. Prove that the k-th column of the matrix $\widetilde{U}$ of (a) contains the coefficients of $u_k(x)$ as in

$$\widetilde{U} = \begin{bmatrix} 1 & u_{0,1} & u_{0,2} & u_{0,3} & \cdots & \cdots & u_{0,n} \\ 0 & 1 & u_{1,2} & u_{1,3} & \cdots & \cdots & u_{1,n} \\ 0 & 0 & 1 & u_{2,3} & \cdots & \cdots & u_{2,n} \\ \vdots & & 0 & 1 & \cdots & \cdots & u_{3,n} \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 & u_{n-1,n} \\ 0 & & & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(c) Assuming now that the moment matrix T has Toeplitz structure derive the so-called Levinson algorithm, that is, an algorithm to compute the columns of $\widetilde{U}$ based on the formula (deduce it) that relates the k-th column u_k of U to its "predecessor" u_{k-1} $(k = 2, 3, \dots, n)$.
Hint: Use the fact (no need to prove it) that Toeplitz moment matrices T have the following property: if

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}$$

then

$$T \begin{bmatrix} x_n^* \\ x_{n-1}^* \\ x_{n-2}^* \\ \vdots \\ x_3^* \\ x_2^* \\ x_1^* \end{bmatrix} = \begin{bmatrix} y_n^* \\ y_{n-1}^* \\ y_{n-2}^* \\ \vdots \\ y_3^* \\ y_2^* \\ y_1^* \end{bmatrix}$$

(d) Prove that the algorithm of (c) uses $O(n^2)$ arithmetic operations.

Q4) Solve (a), (b), (c)

(a) Use the fact that each norm $\| \cdot \|$ on $\mathbb{C}^n$ is uniformly continuous (no need to prove the latter fact, just formulate it as a specific inequality) to prove the following theorem.
All norms on $\mathbb{C}^n$ are equivalent in the following sense. For each pair of norms $p_1(x)$ and $p_2(x)$ there are positive constants m and M satisfying

$$mp_2(x) \leq p_1(x) \leq Mp_2(x)$$

for all x.

(b) Prove that if F is an $n \times n$ matrix with $\|F\| < 1$, then $(I + F)^{-1}$ exists and satisfies

$$\|(I + F)^{-1}\| \leq \frac{1}{1 - \|F\|}.$$

(c) Let A be a nonsingular $n \times n$ matrix, $B = A(I + F)$, $\|F\| < 1$, and x and Δx be defined by

$$Ax = b, \qquad B(x + \Delta x) = b.$$

Use **(b)** to prove that

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\|F\|}{1 - \|F\|}$$

as well as

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{cond(A)}{1 - cond(A)\frac{\|B-A\|}{\|A\|}} \cdot \frac{\|B - A\|}{\|A\|}$$

if

$$cond(A)\frac{\|B-A\|}{\|A\|} < 1.$$

Q5) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

(a) Prove that a positive definite matrix (partitioned as follows:)

$$A = \begin{bmatrix} d_1 & a_{21}^* \\ a_{21} & A_{22} \end{bmatrix}$$

admits a factorization

$$A = \begin{bmatrix} 1 & 0 \\ \frac{1}{d_1}a_{21} & I \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{d_1}a_{21}^* \\ 0 & I \end{bmatrix}$$

with some S, and deduce the formula for S.

(b) Prove that S is also positive definite.

(c) Use the results of (a) and (b) to prove that a positive matrix A admits a factorization

$$A = LDL^*,$$

where L is unit lower triangular (i.e., with 1's on the main diagonal), and D is a diagonal matrix with positive diagonal entries.

(d) Use the result of (c) to prove that a positive matrix A is always invertible and that its inverse is also a positive definite matrix.

(e) Use the result of (c) to prove that all the determinants of leading $k \times k$ submatrices of A are positive ($k = 1, 2, \ldots, n$).