Measure and Integration Prelim, January 2015

Below $(X, \mathcal{F}, \mu)$ is a σ-finite measure space.

1. Prove that the set of continuity points of a function $f : \mathbb{R} \to \mathbb{R}$ is G_δ (a countable intersection of open sets).

2. In this question we assume that $\mu(X) < \infty$.
   Suppose that $(f, f_n : n \in \mathbb{N})$ are real-valued $\mathcal{F}$-measurable functions satisfying

   (a) $f_n \to f$, μ-a.e.

   (b) $\sup_n \int |f_n|^2 d\mu < \infty$.

   Prove that $\lim_{n\to\infty} \int |f_n - f| d\mu = 0$.

3. (a) State and prove Holder's inequality.

   (b) Suppose that $f_1 \in L^2(\mu)$, $f_2 \in L^3(\mu)$ and $f_3 \in L^6(\mu)$. Prove that $f_1 f_2 f_3 \in L^1(\mu)$.

4. Let $p \in [1, \infty]$. Suppose that $\mathcal{G}_0 \subset L^p[0,1]$ (Lebesgue measure) is dense in $L^p[0,1]$, and let

$$\mathcal{G}_1 = \left\{ \int_0^x g(s) ds : g \in \mathcal{G}_0 \right\}.$$

   Determine, according to the value of p, whether $\mathcal{G}_1$ is (necessarily) dense in $L^p[0,1]$. If not, find a counterexample.

5. Let $p \in [1, \infty)$ and let f be a real-valued $\mathcal{F}$-measurable function. Define

$$R_p(x) = x^{p-1} \mu(\{|f| > x\}), x > 0.$$

   Prove:

   (a) If $f \in L^p(\mu)$, then $\lim_{x\to\infty} x R_p(x) = 0$.

   (b) $f \in L^p(\mu)$ if and only if $R_p \in L^1[0, \infty)$ (Lebesuge measure).