Abstract Algebra Prelim Jan. 2016

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. Let F be a field. Prove that $F[x]$ is a Euclidean domain.

2. Let R be a commutative ring with identity. Prove the second isomorphism theorem for R-modules: if M is an R-module and N and P are R-submodules of M then (i) $N+P$ and $N \cap P$ are R-submodules of M and (ii) the quotient R-modules $(N+P)/P$ and $N/(N \cap P)$ are isomorphic. You may use the first isomorphism theorem for R-modules.

3. Let D_{2n} be the dihedral group of order $2n$, with $n \geq 3$.

   (a) Let p be an odd prime and let H be a Sylow p-subgroup of D_{2n}. Prove that H is a normal subgroup and cyclic.

   (b) Writing $2n = 2^e \cdot m$ with m odd and $e \geq 1$, prove that the number of Sylow 2-subgroups of D_{2n} is m.

4. Find (with proof) a product of cyclic groups that is isomorphic to the group

$$(\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z})/\langle(2,6)\rangle.$$

5. For each integer d that's not a perfect square, let R_d be the set of all 2-by-2 matrices of the form

$$\begin{pmatrix} a & bd \\ b & a \end{pmatrix}$$

   with $a, b \in \mathbb{Z}$. Show that R_d is a subring of the ring of integral 2-by-2 matrices $M_2(\mathbb{Z})$ and that R_d is isomorphic to the ring $\mathbb{Z}[\sqrt{d}]$.

6. Give examples as requested, with brief justification.

   (a) A ring R and a map $f\colon R \to R$ such that f is an R-module homomorphism but not a ring homomorphism.

   (b) A commutative ring R and an element $a \neq 0$ or 1 such that $a^2 = a$.

   (c) A non-trivial group with trivial center, $Z(G) = \{e\}$.

   (d) A nonabelian group of order 12 constructed by an explicit semidirect product.