GEOMETRY & TOPOLOGY PRELIM

JANUARY 2016

Problem 1. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = 1$ or $y = -1$. Let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is not Hausdorff.

Problem 2. Suppose $f : X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff. Prove that f is a homeomorphism.

Problem 3. Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to \mathbb{T}^2$ is nullhomotopic.

Problem 4. Let A be a subset of a topological space X. Suppose that $r : X \to A$ is a retraction of X onto A, i.e. r is a continuous map such that the restriction of r to A is the identity map of A.

(1) Show that if X is Hausdorff, then A is a closed subset.
(2) Let $a \in A$. Show that $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.

Problem 5. Let $\mathbb{S}^n$ be an n-dimensional sphere in $\mathbb{R}^{n+1}$ centered at the origin. Suppose $f, g : \mathbb{S}^n \to \mathbb{S}^n$ are continuous maps such that $f(x) \neq -g(x)$ for any $x \in \mathbb{S}^n$. Prove that f and g are homotopic.

Problem 6. Let $k \geq 1$ be an integer. Compute the fundamental groups of the following spaces.

(1) The sphere $\mathbb{S}^2$ with k points removed.
(2) The torus $\mathbb{T}^2$ with k points removed.