Preliminary examination - Numerical Analysis - January 2016

Instructions: Answer all 6 questions. If your answers rely upon known results, you must state those results clearly to receive credit.

1. You are given a matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that for any fixed $y \in \mathbb{R}^4$, there exists a unique $x \in \mathbb{R}^3$ such that $\|y - Ax\| \leq \|y - Az\|$, $\forall z \in \mathbb{R}^3$, where $\|\cdot\|$ is the Euclidean vector norm.

2. Let $f \in \mathcal{C}^\infty[a, b]$, $-\infty < a < b < \infty$, be a real-valued function. A partition $\Delta \equiv \{a = x_0 < x_1 < \ldots < x_n = b\}$ is given, with $n > 1$.

(a) Define the cubic spline interpolant $s(x)$ of data $(x_i, f(x_i))$, $i = 0, 1, \ldots, n$, with side conditions $s''(a) = s''(b) = 0$.

(b) Prove that

$$\int_a^b |s''(x)|^2 \, dx \leq \int_a^b |f''(x)|^2 \, dx.$$

3. Consider solving the linear system $Ax = b$ with $b = (1, 1, 1)^T$ and

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Let $\|\cdot\|$ be a vector norm of your choice. If z is an approximate solution with residual satisfying $\|b - Az\| = 0.01$, derive a numerical upper bound for the relative error $\|x - z\|/\|x\|$.

4. Let $\phi_j : [a, b] \to \mathbb{R}$, $-\infty < a < b < \infty$, for $j = 0, 1, \ldots, n$, $n \geq 1$, be polynomials satisfying

- $\phi_j(x)$ is a polynomial of order precisely j, for each $j = 0, 1, \ldots, n$,
- and the functions are L^2-orthonormal;

$$\int_a^b \phi_j(x)\phi_k(x) \, dx = \delta_{jk}, 0 \leq j, k \leq n.$$

Let $\mathbb{P}^k$ denote the space of all real-valued polynomials of order no more than $k \geq 0$.

(a) Prove that if $p \in \mathbb{P}^k$ with $0 \leq k < j \leq n$, then

$$\int_a^b p(x) \, \phi_j(x) \, dx = 0.$$

(b) Prove that the roots x_i, $i = 1, 2, \ldots, n$, of $\phi_n(x)$ are real, simple, and that $a < x_i < b$.

5. Prove that if $f \in \mathcal{C}^2[a, b]$, $-\infty < a < b < \infty$, is a real-valued function, then there exists $z \in (a, b)$ such that the quadrature error for the trapezoidal rule approximation of $\int_a^b f(x) \, dx$ is precisely $(b - a)^3 f''(z)/12$.

6. Prove that

$$\max_{a \leq x \leq b} |c_n x^n + \ldots + c_1 x + c_0| \geq |c_n| \frac{(b - a)^n}{2^{2n-1}}.$$