Probability Prelim Exam for Actuarial Students
January 2016

1. (10 points) Let X and Y be two random variables with finite expectation.
   (a) (3 points) State the definition that X and Y are independent.
   (b) (7 points) Suppose that $E[XY] = E[X]E[Y]$. Prove that X and Y are independent or disprove it with a counterexample.

2. (10 points) State the Borel-Cantelli Lemma and prove it.

3. (10 points) Let $\{X_1, X_2, \ldots\}$ be a sequence of random variables in a probability space $(\Omega, \mathscr{F}, P)$. Let X be a random variable on the same probability space. Suppose that $\{X_n\}$ converges to X almost surely. Show that for all $\epsilon > 0$, we have

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{|X_i - X| \geq \epsilon\}\right) = 0.$$

4. (10 points) Let τ_1 and τ_2 be two stopping times for a stochastic process $\{X_n\}_{n \geq 0}$. Show that $\min(\tau_1, \tau_2)$ is also a stopping time.

5. (10 points) Let $\{X_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables on a probability space $(\Omega, \mathscr{F}, P)$ with

$$P(X_n = 1) = P(X_n = 0) = \frac{1}{4}, \quad P(X_n = -1) = \frac{1}{2}.$$

   Let a be a positive integer, $S_0 = a$, and

$$S_n = a + \sum_{i=1}^{n} X_i, \quad n \geq 1.$$

   Let $\tau_0 = \inf\{n \geq 0 : S_n = 0\}$. Calculate $P(\tau_0 < \infty)$.

6. (10 points) Let $\{B_t\}_{t \geq 0}$ be a standard Brownian motion. Calculate $E[(B_3 + B_5 + 1)^2]$.