Real Analysis Preliminary Exam, January 2016

Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on $\mathbb{R}$ is denoted by m or dx.

1. (10 points) Compute
$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx$$
and provide full justification for all steps in your reasoning.

2. (15 points) Let K be a compact interval in $\mathbb{R}$ and $f_n : K \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions.
   (a) Suppose that the sequence $\{f_n\}$ converges a.e. on K with respect to m. Show that $\{f_n\}$ converges in m-measure.
   (b) Must the conclusion of part (a) be true if K is not compact? Give a proof or a counterexample.
   (c) Finally, suppose all of the f_n are differentiable and
      i) there is $M < \infty$ such that $\|f_n'\|_\infty \le M$ for all n, and
      ii) for each n there is x_n such that $f_n(x_n) = 0$.

   Prove that there is a subsequence of the f_n that converges uniformly on K to a continuous limit function f.

3. (25 points) Let (X,μ) be a finite measure space and $f : X \to \mathbb{R}$ be measurable. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = \mu(\{x : f(x) < t\})$ and let ν be the associated Lebesgue-Stieltjes measure on $\mathbb{R}$. The goal of this problem is to prove that for a Borel function $g : \mathbb{R} \to \mathbb{R}$, the function $g \circ f$ is integrable on (X,μ) if and only if g is integrable with respect to ν and
$$\int_X g\circ f\,d\mu = \int_{\mathbb{R}} g\,d\nu. \tag{1}$$
One possible approach is to prove the results in the following steps:
   (a) If g is a Borel measurable function on $\mathbb{R}$ prove that $g \circ f$ is measurable on X.
   (b) If g is the characteristic function of an interval $[a,b)$ prove that equation (1) is true.
   (c) Prove that $\{A :$ equation (1) is true for the characteristic function of A$\}$ is a σ-algebra.
   (d) Prove (1) is true when g is a Borel simple function (i.e. a linear combination of characteristic functions of Borel sets).
   (e) Prove (1) is true for any Borel function g, and prove the assertion about integrabilty of $g \circ f$ and g.

4. (15 points) We say that $x \in [0,1]$ has a decimal expansion $x = 0.d_1d_2\dots$ if $x = \sum_{j=1}^{\infty} d_j 10^{-j}$ with $d_j \in \{0,\dots,9\}$. Let
$$E = \{x \in [0,1] : x \text{ has a decimal expansion } x = 0.d_1d_2\dots \text{ in which } d_j \neq 7 \text{ for all } j\}.$$
   (a) Prove that the set of points for which every decimal expansion has $d_1 = 7$ is an open interval.
   (b) Prove that E is compact and has $m(E) = 0$.

   You may use without proof the fact that the decimal expansion is unique unless x is in the countable set having two expansions $x = 0.d_1d_2\dots = 0.d_1'd_2'\dots$ with the property that there is k such that $d_j = d_j'$ for $j < k$, $d_k = d_k' + 1$ and $d_j = 0$, $d_j' = 9$ for $j > k$. (e.g. $0.5 = 0.499\dots$)

   (Hint: First consider the set $E_1 = \{x = 0.d_1d_2\dots : d_1 \neq 7\}$, then $E_2 = \{x = 0.d_1d_2\dots : d_1 \neq 7, d_2 \neq 7\}$, etc.)

5. (15 points) Let μ be a σ-finite measure and let $p \in (1,\infty)$.
   (a) State the Riesz representation theorem for bounded linear functionals on $L^p(\mu)$ for $p \in [1,\infty)$.
   (b) Suppose now that $p \in (1,\infty)$. For each $j \in \mathbb{N}$ let $T_j : L^p(\mu) \to \mathbb{C}$ be a bounded linear functional, and suppose that $\lim_{j\to\infty} T_j g \in \mathbb{C}$ exists for every $g \in L^p(\mu)$. Prove that the mapping $g \mapsto \lim_{j\to\infty} T_j g \in \mathbb{C}$ is a bounded linear functional on $L^p(\mu)$.
   (Hint: If $\{\|T_j\|_{L^p\to L^p} : j \in \mathbb{N}\}$ is unbounded, construct $g \in L^p(\mu)$ such that $\lim T_j g = \infty$.*)*