Applied Math Prelim Jan 2017

1. Let X, Y be normed linear space, $T : X \to Y$ be a linear transformation.

   (a) (10 pts) T is continuous iff T maps every sequence converging to zero into a bounded sequence.

   (b) (10 pts) T is compact iff T maps every sequence that is weakly converging to zero into a sequence which is strongly converging to zero.

   (c) (5 pts) If T is bounded, show that $\|T\| = \sup_{\|x\|_X \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Tx\|_Y$.

2. (20 pts)Find a fundamental solution of operator A defined by $A\varphi = \varphi'' + 3\varphi' + 2\varphi$. (Hint: find fundamental solution $T = \widetilde{f}$ with supp$f \subset [0, \infty)$).

3. (15 pts) Let A be a compact operator on a normed linear space. If $I - A$ is surjective, then it is injective.

4. Let X be a normed linear space.

   (a) (5 pts) Give the defintion of weakly convergence of a sequence $\{x_n\} \subset X$.

   (b) (10 pts) Show that a weakly convergent sequence is bounded.

   (c) (5 pts) Give an example of a sequence that converges weakly to zero but doesn't converge strongly to any point.

5. Let K be a closed convex set in a Hilbert space X. Let $x \in X$ and let $Px \in K$ be the point of K closest to x.

   (a) (10 pts) Prove $\mathcal{R}(x - Px, v - Px) \leq 0$ for all $v \in K$. Here $\mathcal{R}(\cdot, \cdot)$ denotes the real parts of inner product $(\cdot, \cdot)$.

   (b) (10 pts) Show that $\|Px - Py\| \leq \|x - y\|$ for any $x, y \in X$.