Complex Analysis Prelim

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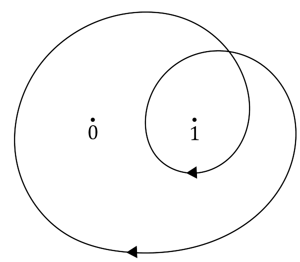
Instructions: Do as many of the following problems as you can. Four completely correct solutions will guarantee a PhD pass. A few completely correct solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill in the gap. You may use any standard theorem from the complex analysis course, identifying it either by name or stating it in full.

Notation and Conventions:

- $\mathbb{C}$ denotes the field of complex numbers and $\mathbb{D}$ denotes the open unit disk
- The terminology *analytic function* and *holomorphic function* may be used interchangeably.



Problems:

1. Compute $\displaystyle\int_\gamma \frac{6z-7}{z^2-z}\,dz$, where γ is the contour displayed on the right.

2. Prove that if f is entire, $f(1)=1$, and $|f(z)| \le e^{1/|z|}$ for all $z \neq 0$, then $f(z)=1$ for all $z \in \mathbb{C}$.

3. Prove that if f is analytic on $\mathbb{D}$, $|f(z)| \le 1$ for all $z \in \mathbb{D}$, and $z_1, \ldots, z_n \in \mathbb{D}$ are zeros of f, then
$$|f(0)| \le \prod_{j=1}^{n} |z_j|.$$
(Convention: if z_j is a zero of order k, then z_j may appear in the list $z_1, \ldots, z_n$ up to k times.)

4. How many zeros (counting multiplicity) does $p(z) = z^6 + 4z^2 - 5$ have in the annulus $\{1 < |z| < 2\}$?

5. Find a one-to-one analytic map from the domain
$$U := \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \setminus \{bi : 0 \le b \le 2\}$$
onto $\mathbb{D}$.

6. Let m be a positive integer. Prove that if f is an entire function and $|f(z)| \le |z|^m$ for all $z \in \mathbb{C}$, then f is a polynomial of degree at most m.

7. Let $\mathcal{F}$ be a family of analytic maps defined on $\mathbb{D}$. Prove that if $M_r := \sup_{f\in\mathcal{F}} \displaystyle\int_{|z|=r} |f(z)|\,|dz| < \infty$ for all $0 < r < 1$, then $\mathcal{F}$ is a normal family.