TOPOLOGY PRELIM, JANUARY 2017

1. On $\mathbb{R}$ we consider the topology τ generated by the basis of all sets of the form (a,b) and all sets of the form $(a,b)\setminus K$, where $K := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.
(a) Prove that $[0,1]$ is not compact in $(\mathbb{R},\tau)$.
(b) Prove that $(\mathbb{R},\tau)$ is connected but it is not path connected.

2. Let X be a set and τ and σ topologies on X so that τ is strictly finer (larger) than σ. Prove the following statements:
(a) If (X,τ) is compact and Hausdorff, then (X,σ) is not Hausdorff.
(b) If (X,σ) is compact and Hausdorff, then (X,τ) is not compact.
Hint: Consider the identity map on X.

3. Define

$$\begin{aligned} A &= \left\{x \in \mathbb{R}^2 : \text{both coordinates of } x \text{ are rational}\right\} \\ B &= \left\{x \in \mathbb{R}^2 : \text{ at least one coordinate of } x \text{ is rational}\right\}. \end{aligned}$$

Show that $\mathbb{R}^2 \setminus A$ is connected and $\mathbb{R}^2 \setminus B$ is not connected.

4. Let X be a topological space and $f, g : X \to \mathbb{S}^2$ two continuous maps. Show that if for every $x \in X$ the points $f(x)$ and $g(x)$ on $\mathbb{S}^2$ are not antipodal to each other, then f and g are homotopic.

5. Compute the fundamental groups of the following spaces:
(a) The torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.
(b) The pinched torus $\mathbb{S}^1 \times \mathbb{S}^1 \diagup \mathbb{S}^1 \times \{1\}$.

6. Assume $p : \tilde{X} \to X$ is a covering and both X and $\tilde{X}$ are path connected. Assume A is a path connected subset of X so that $i_* : \pi_1(A,a) \to \pi_1(X,a)$ is onto, for some $a \in A$, where $i : A \to X$ is the inclusion map. Prove that $p^{-1}(A)$ is path connected as well.