Qualifying Exam

1. Show that if $v \in \mathbb{R}^N$ and $v^T v = 1$, then the matrix $Q = I - 2vv^T$ is both symmetric and orthogonal.

2. Consider a quadrature of the form

$$\int_{-1}^{1} |x| f(x)\ dx = \frac{1}{4}(f(-1) + 2f(0) + f(1)).$$

   Show that it is exact for any polynomial $f(x)$ of degree at most 3.

3. Consider the Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 1, 2, \ldots,$$

   for finding the root of the equation $f(x) = 0$. Assume that $\bar{x}$ is a root of multiplicity m, i.e., $f(x) = (x - \bar{x})^m g(x)$, where $m > 1$ is an integer and $g(x)$ is a smooth function with $g(\bar{x}) \neq 0$ and the Newton's method converges to $\bar{x}$. Show that Newton's method must converge to $\bar{x}$ only linearly. How would you modify the method to obtain quadratic convergence?

4. Consider a matrix A and it inverse A^{-1}

$$A = \begin{pmatrix} -0.4 & 1.0 & -0.8 \\ 1.2 & -2.0 & 1.4 \\ -0.6 & 1.0 & -0.2 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$$

   (a) What is $\|A\|_1$ and $\|A\|_\infty$?

   (b) What is the condition number of A in 1-norm?

   (c) Suppose $Ax = b$ and $(A + E)\hat{x} = b$, where $\|E\|_1 \leq 0.01$. Give a bound on the relative difference between the two solutions in 1-norm.

5. The barycentric form of Lagrange's interpolation takes the form

$$p_n(x) = \frac{\sum_{j=0}^{n} w_j f(x_j)/(x - x_j)}{\sum_{j=0}^{n} w_j/(x - x_j)},$$

   where

$$w_j = \frac{1}{\Psi_n'(x_j)} \quad \text{with} \quad \Psi_n(x) = \prod_{j=0}^{n} (x - x_j).$$

   Verify that the above formula indeed produces the unique interpolation polynomial.

6. Suppose that $g : [a, b] \to [a, b]$ is continuous on interval $[a, b]$ and is a contraction, i.e. there exists a constant $L \in (0, 1)$ such that

$$|g(x) - g(y)| \leq L|x - y|, \quad \forall x, y \in [a, b].$$

   Prove that there exists a unique fixed point in $[a, b]$ and that the fixed point iteration $x_{n+1} = g(x_n)$ converges to the fixed point for any $x_0 \in [a, b]$. Also, prove that the error is reduced by a factor of at least L from each iteration to the next.