Probability Prelim Exam for Actuarial Students
January 2017

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

1. (10 points) Let $\{A_n : n \geq 1\}$ be a sequence of events in $(\Omega, \mathscr{F}, P)$. Show that

$$P(\liminf A_n) \leq \liminf P(A_n) \leq \limsup P(A_n) \leq P(\limsup A_n),$$

where $\liminf A_n$ and $\limsup A_n$ are defined as

$$\liminf A_n = \bigcup_{i \geq 1} \left(\bigcap_{j \geq i} A_j \right), \quad \limsup A_n = \bigcap_{i \geq 1} \left(\bigcup_{j \geq i} A_j \right).$$

2. (10 points) Jensen's inequality.
   (a) (5 points) State Jensen's inequality.
   (b) (5 points) Prove Jensen's inequality.

3. (10 points) Let (X, Y) be bivariate normally distributed with mean 0. The joint probability density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} \right] \right).$$

Show that X and Y are independent if and only if

$$E(XY) = 0.$$

4. (10 points) Let $\{Z_n : n \geq 1\}$ be a sequence of random variables on $(\Omega, \mathscr{F}, P)$. Suppose that $Z_n \to Z$ almost surely, where Z is a random variable on the same probability space. Show that $Z_n \to Z$ in probability.

5. (10 points) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with finite expectation. Calculate $E[X_1 | X_1 + X_2 + \cdots + X_n]$.

6. (10 points) Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed random variables with the following distribution:

$$P(X_1 = 1) = \frac{2}{3}, \quad P(X_1 = -1) = \frac{1}{3}.$$

Let $S_0 = 0$ and $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$.

(a) (5 points) Show that $\{Z_n : n \geq 0\}$ is a martingale, where $Z_n = 2^{-S_n}$ for $n \geq 0$.

(b) (5 points) Let $\tau = \inf\{n \geq 0 : |S_n| = M\}$ for some integer $M > 0$. Calculate $P(S_\tau = M)$.

7. (10 points) Let $\{B_t : t \geq 0\}$ be a standard Brownian motion. Let $W_0 = 0$ and

$$W_t = tB_{\frac{1}{t}}, \quad t > 0.$$

Show that $\{W_t : t \geq 0\}$ is a Brownian motion.