Abstract Algebra Prelim Jan. 2018

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) Prove every group of order $2p$, where p is an odd prime, is either cyclic or is isomorphic to the dihedral group of order $2p$.

2. (**10 pts**)

   (a) (**3 pts**) If G is a group with an abelian normal subgroup N of index 2 and $a \in G - N$, prove a subgroup H of N is normal in G if $aHa^{-1} = H$.

   (b) (**4 pts**) Let $G = (\mathbf{Z}/3\mathbf{Z})^2 \rtimes_\varphi \mathbf{Z}/2\mathbf{Z}$, where $\varphi : \mathbf{Z}/2\mathbf{Z} \to \operatorname{Aut}((\mathbf{Z}/3\mathbf{Z})^2)$ is the action of $\mathbf{Z}/2\mathbf{Z}$ on $(\mathbf{Z}/3\mathbf{Z})^2$ that sends the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ to the automorphism $(x, y) \mapsto (y, x)$ of $(\mathbf{Z}/3\mathbf{Z})^2$. Use part (a) to show $H = \langle(1, 2)\rangle \times \{0\} = \{(1, 2), (2, 1), (0, 0)\} \times \{0\}$ is a normal subgroup of G.

   (c) (**3 pts**) With G and H as in part (b), determine whether G/H is abelian.

3. (**10 pts**)

   (a) (**4 pts**) Prove the direct product ring $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$ (componentwise operations) and the quotient ring $\mathbf{Z}[x]/(x^2)$ are not isomorphic.

   (b) (**3 pts**) Prove $\mathbf{Z}^2 \cong \mathbf{Z}[x]/(x^2 - x)$ as rings.

   (c) (**3 pts**) For integers $c \geq 2$, prove $\mathbf{Z}^2 \ncong \mathbf{Z}[x]/(x^2 - cx)$ as rings. (Hint: for a ring A, consider A/pA for a suitable prime number p.)

4. (**10 pts**) Let $G = \mathbf{Z}/24\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$.

   (a) (**4 pts**) What is the order of the element $(10, 3, 2)$ in G? (Be sure this is right to solve part b.)

   (b) (**6 pts**) Consider the quotient group $H = G/\langle(10, 3, 2)\rangle$. Determine a direct product of cyclic groups that is isomorphic to H.

5. (**10 pts**) Let R be a commutative ring with identity. Prove that R has a unique maximal ideal if and only if for all x and y in R satisfying $x + y = 1$, x or y is a unit in R.

6. (**10 pts**) Give examples as requested, with justification.

   (a) (**2.5 pts**) An integral domain that is not a PID.

   (b) (**2.5 pts**) Find a permutation $\pi \in S_6$ such that $\pi(12)(456)\pi^{-1} = (36)(154)$.

   (c) (**2.5 pts**) An element of an integral domain that is irreducible but not prime.

   (d) (**2.5 pts**) A polynomial $f(x)$ in $(\mathbf{Z}/2\mathbf{Z})[x]$ such that the quotient ring $(\mathbf{Z}/2\mathbf{Z})[x]/(f(x))$ is a field of order 8.