TOPOLOGY PRELIM, JANUARY 2018

Convention

- Let A and B be two sets. We denote $A \setminus B = \{x \in A; x \notin B\}$.
- Unless otherwise indicated, the space $\mathbb{R}^n$ and its subsets given below are endowed with the standard topology.

1. Let X be a topological space. For any subset A of X, is it always true that $X \setminus \overline{A} = \operatorname{int}(X \setminus A)$? Prove your assertion. (Here $\overline{A}$ denotes the closure of A and $\operatorname{int}(B)$ denotes the set of interior points of a set B.)

2. Let $B = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$, $p = (1/2, 0)$, and $q = (-1/2, 0)$. Denote $M = B \setminus \{p, q\}$. Is M homotopic to the boundary of B? Prove your assertion.

3. Let X be the union of the unit sphere $S^2 \equiv \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ with the two line segments
$$\{(0, y, 0); |y| \leq 1\} \cup \{(0, 0, z); |z| \leq 1\}.$$
Compute the fundamental group of X based at $(0, 1, 0)$.

4. Let E be a subset of a topological space Y. Suppose that $f : Y \to E$ is a continuous map such that $f(x) = x$ for all $x \in E$. Show that if Y is Hausdorff, then E is a closed subset of Y.

5. Let $\mathrm{Mat}_2(\mathbb{R})$ be the set of 2×2 real matrices with the topology obtained by regarding $\mathrm{Mat}_2(\mathbb{R})$ as $\mathbb{R}^4$. Let
$$\mathrm{SO}(2) = \{A \in \mathrm{Mat}_2(\mathbb{R}); A^T A = I_2, \det A = 1\}$$
where A^T denotes the transpose of A, and I_2 is the 2×2 identity matrix.
   (i) Show that $\mathrm{SO}(2)$ is compact.
   (ii) Is $\mathrm{SO}(2)$ connected? Prove your assertion.

6. Find a simply-connected covering space for the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$. Justify your reasoning. (The space $\mathbb{RP}^2$ is the quotient space of the unit sphere S^2 obtained by identifying the antipodal points.)