Probability Prelim Exam for Actuarial Students
January 2018

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

1. (10 points) Give an example of events A, B, and C that satisfy the following conditions:
   (a). $P(A) \in (0,1)$, $P(B) \in (0,1)$, and $P(B) \in (0,1)$.
   (b). $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, and $P(A \cap B \cap C) = P(A)P(B)P(C)$.
   (c). $P(B \cap C) \neq P(B)P(C)$

2. (10 points) Let $\{E_n\}_{n\geq 1}$ be a sequence of events in a probability space $(\Omega, \mathscr{F}, P)$. Suppose that
$$\liminf_{n\to\infty} P(E_n) = 0$$
and
$$\sum_{n=1}^{\infty} P(E_{n+1} \cap E_n^c) < \infty.$$
show that $P(\limsup E_n) = 0$.

3. (10 points) Let X be a random variable in L^1. Prove that
$$E[X] = \int_0^{\infty} [P(X > t) - P(X < -t)] \, \mathrm{d}t.$$

4. (10 points) Let X be a random variable that has finite mean m and finite variance σ^2. Show that for any $\alpha > 0$,
$$P(X \geq \alpha + m) \leq \frac{\sigma^2}{\sigma^2 + \alpha^2}.$$

5. (10 points) Suppose that $X_1, X_2, \ldots$ are IID with distribution function F. Define
$$\hat{F}_n(x) = \frac{1}{n} \sum_{j \leq n} \mathbf{1}_{\{X_i \leq x\}}.$$
Prove the following:
   (a). $\hat{F}_n(x) \to F(x)$ for **all** x, a.s.
   (b). If $X_1, X_2, \ldots$ are in L^1, then
$$\lim_{n\to\infty} E\left[\int_0^{\infty} (\hat{F}_n(x) - F(x))^2 \mathrm{d}x\right] = 0$$

6. (10 points) Let $\{X_n\}_{n\geq 0}$ be a simple symmetric random walk with $X_0 = 10$. Let $\tau = \min\{n \geq 1 : X_n = 0\}$. Calculate the following quantities
   (a) (3 points) $E[X_{100}]$.
   (b) (3 points) $E[X_\tau]$.
   (c) (4 points) $E[X_{\min\{n,\tau\}}]$.
7. (10 points) Let $(B_t : t \geq 0)$ be standard Brownian motion. For each $i = 1, 2, \ldots,$ let $Z_i = 1$ if $B_i > B_{i-1}$ and zero otherwise, and let $T = \inf\{n : \sum_{i\leq n} Z_i = 10\}$. Find the expectation of B_T.