Real Analysis Preliminary Exam, January 2018

Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Questions have equal weight, but one complete and correct question is worth more than two half-complete questions

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1. (a) State the Arzelà-Ascoli theorem for a compact metric space. (Do not prove this.)
   (b) Define what it means for a sequence of functions f_n to be equicontinuous.
   (c) Suppose f_n is a sequence of continuous functions that is Cauchy in the uniform norm on a compact metric space X. Show that the sequence f_n is equicontinuous.

2. Let f be a non-negative Lebesgue measurable function on $[0, 1]$ and let

$$A = \{(x, y) : 0 \leq y \leq f(x),\ x \in [0, 1]\}.$$

   Is A measurable with respect to 2-dimensional Lebesgue measure? If so, what is its measure? If not, what additional assumption(s) would you need to ensure it is measurable and compute its measure?

3. Let $1 \leq p < q < \infty$ and $(X, \mathcal{F}, \mu)$ be a measure space.
   (a) Prove that $L^p(X, \mu) \not\subseteq L^q(X, \mu)$ if X contains sets of arbitrarily small positive measure.
   (b) Prove that $L^q(X, \mu) \not\subseteq L^p(X, \mu)$ if X contains sets of arbitrarily large finite measure.

4. (a) Suppose f_n and f are in $L^1(X, \mu)$ and that $f_n \to f$ a.e. Show that $f_n \to f$ in L^1 if and only if $\|f_n\|_1 \to \|f\|_1$.
   (Hint: For one direction it may help to consider the sequence $|f| + |f_n| - |f - f_n|$.)
   (b) Give an example in which L^1 convergence does not imply a.e. convergence.

5. If $\mu \ll \nu$ and $\nu \ll \mu$ are finite measures on a measurable space $(X, \mathcal{F})$ show that the Radon-Nikodym derivatives satisfy $\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} = 1$ μ-a.e.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
   (a) If f also has the property that for each $y \in \mathbb{R}$ there is at most one value of x with $f(x) = y$ prove that f is differentiable almost everywhere on $\mathbb{R}$.
   (b) If, instead, f has the property that for each $y \in \mathbb{R}$ there are at most two values of x with $f(x) = y$ prove that f is differentiable almost everywhere on $\mathbb{R}$.