Justify all your steps. You may use any results that you know unless the question says otherwise, but don’t invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (10 pts)
   (a) (7 pts) For \( n \geq 3 \), determine with proof the conjugacy classes of the dihedral group of order \( 2n \). (Hint: Separately consider even \( n \) and odd \( n \).)
   (b) (3 pts) Let \( C_n \) be the number of conjugacy classes in the dihedral group of order \( 2n \). Compute \( \lim_{n \to \infty} \frac{C_n}{n} \).

2. (10 pts) Let \( p \) the smallest prime dividing the order of a finite group \( G \). Prove that if \( H \) is a subgroup of \( G \) with index \( p \) then \( H \) is a normal subgroup. (Hint: Look at the left multiplication action of \( G \) on the left cosets of \( H \).)

3. (10 pts) View \( \mathbb{Q} \) and \( \mathbb{Z} \) as additive groups. For \( a \in \mathbb{Z} \), set \( \varphi_a : \mathbb{Q} \to \mathbb{Q} \) by \( \varphi_a(t) = 2^a t \).
   (a) (4 pts) Show that \( \varphi_a \) is an automorphism of (the additive group) \( \mathbb{Q} \) for each \( a \in \mathbb{Z} \) and show \( \varphi : \mathbb{Z} \to \text{Aut}(\mathbb{Q}) \) given by \( a \mapsto \varphi_a \) is a homomorphism of groups.
   (b) (4 pts) Set \( G = \mathbb{Q} \rtimes_\varphi \mathbb{Z} \), a semi-direct product. In \( G \) let \( H = \{(m,0) : m \in \mathbb{Z}\} \) and \( x = (0,1) \). Prove that \( xHx^{-1} \subset H \).
   (c) (2 pts) Show that \( x = (0,1) \) is not an element of the normalizer \( N_G(H) \) of \( H \) in \( G \).

4. (10 pts)
   (a) (4 pts) Define a Euclidean domain and prove all ideals in a Euclidean domain are principal.
   (b) (4 pts) Prove \( F[X] \) is a Euclidean domain when \( F \) is a field.
   (c) (2 pts) Prove \( \mathbb{Z}[X] \) is not a Euclidean domain.

5. (10 pts)
   (a) (2 pts) For a commutative ring \( R \) and \( R \)-module \( M \), define what it means to say \( M \) is a cyclic \( R \)-module.
   (b) For any matrix \( A \in M_n(R) \), we can make \( \mathbb{R}^n \) into an \( R[t] \)-module by declaring that for any polynomial \( f(t) = c_0 + c_1 t + \cdots + c_d t^d \) in \( R[t] \) and vector \( v \) in \( \mathbb{R}^n \), \( f(t)v = f(A)v = (c_0 I + c_1 A + \cdots + c_d A^d)v \).
   Determine, with explanation, whether \( \mathbb{R}^n \) is a cyclic \( R[t] \)-module for each of the following choices of \( A \). If it is a cyclic \( R[t] \)-module, then find an \( R[t] \)-generator:
   i. (4 pts) \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) on \( \mathbb{R}^2 \),
   ii. (4 pts) \( A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \) on \( \mathbb{R}^3 \).

6. (10 pts) Give examples as requested, with justification.
   (a) (2.5 pts) A group isomorphism from \( (\mathbb{Z}/7\mathbb{Z})^\times \) to \( (\mathbb{Z}/9\mathbb{Z})^\times \).
   (b) (2.5 pts) A cyclic group with 20 generators.
   (c) (2.5 pts) A unit in \( \mathbb{Z}[\sqrt{11}] \) other than \( \pm 1 \).
   (d) (2.5 pts) A prime element of \( \mathbb{Z}[i] \).