Complex Analysis Prelim

January 2019

Notation and conventions:

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.
- The terminology *analytic* function and *holomorphic* function may be used interchangeably.

Problem 1. How many solutions (counted with multiplicity) does the equation $z^6+5z^3+1 = 0$ have in the unit disk $\mathbb{D}$?

Problem 2. Let f be a holomorphic map of the unit disk $\mathbb{D}$ into itself. Suppose f is not the identity map. Can f have two or more fixed points? Prove your assertion. (Recall $w \in \mathbb{D}$ is a fixed point of f if $f(w) = w$.)

Problem 3. Prove or disprove that there exists a holomorphic function $f(z)$ defined on the punctured disk $\mathbb{D} \setminus \{0\}$ such that

$$\lim_{z\to 0} zf(z) = 0 \quad \text{and} \quad \lim_{z\to 0} |f(z)| = \infty.$$

Problem 4. Find a one-to-one conformal map from $U = \{z \in \mathbb{C} : |z| > 1 \text{ and } \mathrm{Im}(z) > 0\}$ onto the unit disk $\mathbb{D}$.

Problem 5. Suppose f is a non-constant holomorphic function on $\mathbb{D}$. Suppose $|f|$ is constant on the circle $|z| = \frac{1}{2}$. Show that f has at least one zero in $\Omega = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$.

Problem 6. Let a be a positive real number. Compute

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\, dx.$$

Problem 7. Is there a one-to-one conformal map from the punctured disk $\mathbb{D} \setminus \{0\}$ onto the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$? Prove your assertion.