Topology Prelim, January 2019

1. Let X be the subset $(\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2$. Define an equivalence relation on X by declaring $(x, 0) \sim (x, 1)$ if $x \neq 0$. Show that the quotient space $X \backslash \sim$ is not Hausdorff.

2. Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be **locally finite** if each point of X has a neighborhood that intersects at most finitely many of the sets in $\mathcal{A}$. Show that if $\mathcal{A}$ is a locally finite collection of subset of X, then

$$\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \bar{A}.$$

3. Let $X = (\mathbb{R}^2 \times \{0\}) \cup \{(0, y, z) | y^2 + z^2 = 1, z \geq 0\} \cup \{(x, 0, z) | x^2 + z^2 = 1, z \geq 0\}$. Compute the fundamental group of X based at $(0, 0, 0)$.

4. Let $\mathbb{P}^2$ denote the projective plane. Prove that any continuous map $f : \mathbb{P}^2 \to \mathbb{T}^2$ is nullhomotopic, i.e. homotopic to a constant map.

5. Let $U = \mathbb{R}^2 \backslash S = \{x \in \mathbb{R}^2 | x \notin S\}$, where $S \subset \mathbb{R}^2$ is a countable set. Is U path-connected? Justify your answer.

6. Let X be a topological space and $q : \mathbb{R}^2 \to X$ be a covering map. Let $B = \{(x, y) | x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ and let K be a compact subset of X. Suppose $q : \mathbb{R}^2 \backslash B \to X \backslash K$ is a homeomorphism. Show that X is homeomorphic to $\mathbb{R}^2$.