Preliminary Examination - Numerical Analysis - January, 2019

Instructions: Solve 6 out of 8 problems. Any results that your responses rely upon must be stated clearly.

1. Given any real-valued function $f = f(x)$ that is continuous for all $x \in [-1, 1]$, let $I(f) = \int_{-1}^{1} f(x)\,dx$. Define a quadrature approximation by

$$\tilde{I}(f) = w_{-1} f(-1) + w_0 f(0) + w_1 f(1).$$

Here, the weights w_j, $j = -1, 0, 1$, are real numbers such that $I(f) = \tilde{I}(f)$ whenever f is a polynomial of degree four or less with $f(0) = f'(0)$ and $f(1) = f'(1)$.
(a) Prove that the weights w_j, $j = -1, 0, 1$, exist and are unique.
(b) Prove that there exists a positive constant $C > 0$ such that for all $f \in \mathcal{C}^5[-1, 1]$ with $f(0) = f'(0)$ and $f(1) = f'(1)$,

$$\tilde{I}(f) - I(f) = C f^{(5)}(\xi),$$

for some $\xi \in (-1, 1)$, where ξ may depend on f, but C does not.

2. Given a space of real-valued functions $X = \{f \in \mathcal{C}^2[-1, 1] \mid f'(-1) = f'(1) = 0\}$, define the subset $Y = \{f \in X \mid f(-1) = 0,\ f(0) = 1, \text{ and } f(1) = 4/3\}$. Prove that

$$\inf_{f \in Y} \int_{-1}^{1} |f''(x)|^2 \, dx = \frac{16}{3}.$$

3. Let $n > 0$ be an integer. For each $j = 0, 1, \ldots, n$, let $\phi_j(x)$ be a real-valued polynomial of degree n or less. Assume that for any integers $0 \le i \le n$ and $0 \le j \le n$,

$$\int_0^1 \phi_i(x)\phi_j(x)\,dx = \begin{cases} 2, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Prove that the polynomials ϕ_j, $0 \le j \le n$, are linearly independent.
(b) Given a continuous function $f : [0, 1] \to \mathbb{R}$, provide the explicit formulas for the values c_j, $0 \le j \le n$, such that $p(x) = \sum_{j=0}^{n} c_j \phi_j(x)$ satisfies

$$\int_0^1 (f(x) - p(x))^2 \, dx \le \int_0^1 (f(x) - q(x))^2 \, dx, \tag{1}$$

for all real-valued polynomials $q(x)$ of degree n or less. Subsequently, prove that (1) holds.

4. Given the matrix $\mathbf{A}$ below, calculate an upper-triangular matrix $\mathbf{R}$ for a QR-factorization $\mathbf{A} = \mathbf{QR}$ by using Householder matrices. Do not calculate $\mathbf{Q}$, but formulas for each Householder matrix must be shown, specifying numerical values for all quantities in the formulas.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 1 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

5. Given any matrix $\mathbf{S}$, denote by $\mathbf{S}^H$ the result of replacing each entry of $\mathbf{S}$ by its complex conjugate, then transposing the resulting matrix. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices that satisfy $\mathbf{U}^H\mathbf{A}\mathbf{V} = \mathbf{B}$, where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices. Denote by $\mathbf{A}^+$ and $\mathbf{B}^+$ the pseudoinverses of $\mathbf{A}$ and $\mathbf{B}$, respectively. Prove that $\mathbf{V}^H\mathbf{A}^+\mathbf{U} = \mathbf{B}^+$.

6. Denote by $\|\cdot\|_2$ the Euclidean vector norm on $\mathbb{C}^n$, and let M_n be the space of all complex, square matrices with n rows. Given any $\mathbf{A} \in M_n$, denote the induced matrix norm by

$$\|\mathbf{A}\| = \max_{\|\vec{x}\|_2=1} \|\mathbf{A}\vec{x}\|_2.$$

Next, let $\mathbf{A} \in M_n$ be a fixed Householder matrix, and assume $\mathbf{B} \in M_n$ is given such that
$\|\mathbf{B} - \mathbf{A}\| = 1/3$.
(a) Prove that $\mathbf{B}$ is nonsingular.
(b) If $\mathbf{A}\vec{x} = \mathbf{B}\vec{y}$, prove that $2\|\vec{x} - \vec{y}\|_2 \leq \|\vec{x}\|_2$.

7. Given any real-valued function $f = f(x)$ that is continuous for all $x \in [0, 1]$ and differentiable at $x = 0$, let $\tilde{I}(f)$ denote the quadrature approximation

$$\tilde{I}(f) = \frac{1}{12}\left(5f(0) + f'(0) + 4f(1/2) + 3f(1)\right),$$

with associated quadrature error, say $R(f)$, defined by

$$R(f) = \tilde{I}(f) - \int_0^1 f(x)\,dx.$$

Given any $f \in \mathcal{C}^3[0, 1]$, prove that there exists some $y \in (0, 1)$ such that

$$R(f) = \frac{1}{144} f^{(3)}(y).$$

8. Given any real-valued function $f \in \mathcal{C}[-1, 1]$, let $I(f) = \int_{-1}^{1} w(x)f(x)\,dx$, where $w(x) = x^2$. Derive a quadrature rule that calculates $I(f)$ exactly whenever f is a polynomial of degree three or less.