Probability Prelim Exam for Actuarial Students
January 2019

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

1. (10 points) Let $\mathscr{F}$ be the smallest class of subsets of $\mathbb{N} = \{1, 2, \dots\}$ with the property that $A \in \mathcal{F}$ if and only if A or its complement A^c are finite (the emptyset is finite).
   (a) (3 points) Find a subset of $\mathbb{N}$ not in $\mathcal{F}$.
   (b) (7 points) Show that the smallest σ-algebra containing $\mathcal{F}$ is the power set of $\mathbb{N}$, that is all subsets of $\mathbb{N}$.

2. (10 points) Suppose that $(X_n : n \in \mathbb{N})$ are positive RVs satisfying $E[X_n] = 1$. Use the Borel-Cantelli Lemma to show that $\limsup_{n\to\infty} \frac{\ln X_n}{n} \leq 0$, a.s.

3. (10 points) Let X be a random variable on a probability space $(\Omega, \mathscr{F}, P)$. Suppose that X takes only on nonnegative integer values. Show that

$$E[X] = \sum_{k=0}^{\infty} P\{X > k\}.$$

4. (10 points) Let X be standard normal. Compute $E[|X|]$.

5. (10 points) Let $X_1, X_2, \dots$ be a sequence of random variables, with $E[X_n] = 8$ and $Var[X_n] = 1/\sqrt{n}$ for each $n = 1, 2, \dots$. Prove or disprove that $\{X_n\}$ must converge to 8 in probability.

6. (10 points) Let $Z_1, Z_2, \dots$ be a sequence of independent and identically distributed random variables that follow the following distribution:

$$P(Z_1 = 1) = \frac{3}{4}, \quad P(Z_1 = -1) = \frac{1}{4}.$$

   Let $X_0 = -10$ and $X_n = X_0 + Z_1 + \cdots + Z_n$ for $n \geq 1$. Let τ be a stopping time defined as $\tau = \min\{n \geq 1 : X_n = 0\}$. Compute $E[\tau]$ and $E[X_\tau]$.

7. (10 points) Let $(B_t : t \geq 0)$ be standard Brownian motion and $\sigma > 0$. Let

$$X_t = \exp\left(\sigma B_t - \frac{\sigma^2}{2} t\right), \quad t \geq 0.$$

   Show that $\{X_t\}$ is a martingale.