Abstract Algebra Prelim — Aug. 2019

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) For a ring R, write $\mathrm{GL}_3(R)$ for the group of 3×3 matrices with entries in R and determinant in the units $R^\times$.

   (a) (**5 pts**) Give, with reasoning, a matrix in $\mathrm{GL}_3(\mathbf{Z})$ with first row $\begin{pmatrix} 6 & 10 & 15 \end{pmatrix}$.

   (b) (**5 pts**) Let $\mathbf{Z}[x]$ be the polynomial ring with coefficients in $\mathbf{Z}$. Show that no matrix in $\mathrm{GL}_3(\mathbf{Z}[x])$ has first row $\begin{pmatrix} 6 & 2x & 3x \end{pmatrix}$.

2. (**10 pts**)

   (a) (**2 pts**) For prime p, define a p-Sylow subgroup of a finite group G.

   (b) (**4 pts**) Prove that if a p-group H acts on a finite set X then $\#X \equiv \#\mathrm{Fix}_H(X) \bmod p$, where $\mathrm{Fix}_H(X)$ is the set of points in X fixed by all of H.

   (c) (**4 pts**) For each prime p, prove that if P and Q are p-Sylow subgroups of a finite group G then P and Q are conjugate in G. (That is, prove the second part of the Sylow theorems.) You may use part (b).

3. (**10 pts**) Let F be a field.

   (a) (**5 pts**) Prove that if $f(X) \neq 0$ in $F[X]$ then it has at most $\deg f$ different roots in F.

   (b) (**5 pts**) If $f(X_1, \ldots, X_n) \in F[X_1, \ldots, X_n]$ where F is infinite and $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in F$ then prove $f = 0$ in $F[X_1, \ldots, X_n]$. You may use part (a).

4. (**10 pts**) Let R be a nonzero commutative ring with identity. A *simple* R-module is a *nonzero* R-module M whose only submodules are $\{0\}$ and M. Let A, B, and C all be simple R-modules.

   (a) (**4 pts**) Show that an R-module homomorphism $f\colon A \to B$ is either 0 or an isomorphism.

   (b) (**6 pts**) Suppose that $A \oplus C \cong B \oplus C$ as R-modules. Prove that $A \cong B$ as R-modules. You may use part (a).

   Caution! Part (b) can fail for modules that are not all simple. For some rings R there is an R-module M such that $M \oplus R \cong R^2 \oplus R$ and $M \not\cong R^2$.

5. (**10 pts**) Let R be a commutative ring with identity.

   (a) (**2 pts**) Define what it means for R to be a principal ideal domain.

   (b) (**8 pts**) Prove that if R is a principal ideal domain, then every nonzero prime ideal in R is a maximal ideal.

6. (**10 pts**) Give examples as requested, with justification.

   (a) (**2.5 pts**) A noncyclic group that is *not* isomorphic to a semidirect product of nontrivial groups.

   (b) (**2.5 pts**) A prime p such that the ideal $(p, x^2 - 3)$ in $\mathbf{Z}[x]$ is maximal.

   (c) (**2.5 pts**) A UFD that is not a Euclidean domain.

   (d) (**2.5 pts**) A cyclic $\mathbf{R}[T]$-module that is 2-dimensional as a real vector space.