Applied Math Prelim August 2019

1. Let X be a normed linear space.
   (a) Define weak convergence in a normed linear space and show that weak limit of a sequence is unique.
   (b) Prove that a weakly convergent sequence is bounded.
   (c) Give an example of a weak convergent sequence which is not strongly convergent.

2. Given the Sturm-Liouville operator $Au = u'' + u$ with $u(0) = u'(1) = 0$.
   (a) Find an orthonormal basis of $L^2[0,1]$ via operator A.
   (b) Explain the theory behind your method.

3. Let $f : D \to Y$ be a mapping from an open set D in a normed linear space X to another normed linear space Y.
   (a) State the definition of Fréchet derivative of f.
   (b) For $f : C^1[0,1] \to \mathbb{R}$ defined by
   $$f(u) = \int_0^1 u(x)\sqrt{1 + (u'(x))^2}\,dx,$$
   find its Fréchet derivative.

4. Show that $u(x) = -\frac{1}{2\pi}\ln|x|$ is the fundamental solution to operator $-\Delta$ in $\mathbb{R}^2$. (i.e. $-\Delta u = \delta_0$)

5. Given a system of differential equations
   $$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \tag{1}$$
   with initial conditions $x(0) = a$ and $y(0) = b$.
   (a) State an existence and uniqueness theorem for (1) under the assumption that f, g and all their partial derivatives are continuous.
   (b) For the system
   $$\begin{cases} x' = x(1 - 2x - y) \\ y' = y(2 - x - y) \end{cases}$$
   with $x(0) = y(0) = 0.5$, can either $x(t)$ or $y(t)$ become zero in finite time? Explain your answer.